

**Non-perturbative simulation  
of chiral fermions**

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# Non-perturbative ( lattice ) regularization of chiral gauge theories

## *Weyl fermions from Dirac fermions*

Basic idea is in the continuum:

How to regulate a Weyl fermion using Pauli-Villars fields?

S.A. Frolov and A.A. Slavnov, Phys. Lett. B309 (1993) 344.

D.B. Kaplan, Phys. Lett. B288 (1992) 342.

$$\mathcal{L}_\psi = \bar{\psi} \gamma_\mu D_\mu \psi + \bar{\psi} (P_L \mathcal{M} + P_R \mathcal{M}^\dagger) \psi$$

$\bar{\psi}$  and  $\psi$  are Dirac fermions.

$\mathcal{M}$  is an infinite mass matrix. It has a single zero mode but its adjoint has no zero modes.

Kaplan's choice:

$$\mathcal{M} = \partial_s + M\epsilon(s)$$

## The Overlap formalism

R. Narayanan and H. Neuberger, Nucl. Phys. B443 (1995) 305  
(hep-th/9411108).

The determinant of the chiral Dirac Operator

$$C = \sigma_\mu (\partial_\mu + iA_\mu)$$

can be realized on the lattice as an overlap of two many body states.

$$\det C = \langle 0 - | 0 + \rangle$$

where  $|0\pm\rangle$  are many body ground states of  $a^\dagger H(m)a$  and  $a^\dagger \gamma_5 a$  respectively.

$a^\dagger$  and  $a$  are canonical fermion creation and destruction operators.

$\gamma_5 H(m)$  is a realization of a massive Dirac operator on the lattice with the mass set to a value less than zero on the lattice.

A natural choice is  $H_w(m) = \gamma_5 D_w(m)$  where  $D_w(m)$  is the massive Wilson Dirac operator with  $m < m_c$ .

**Why is it *natural* to write the determinant of  $C$  as an overlap of two many body states?**

1)  $C$  is an operator that maps two different spaces, namely spinors under the  $(0,1/2)$  representation to  $(1/2,0)$  representation.

2)  $C$  does not have an eigenvalue problem.

3) Determinant of a generic operator is a map that takes the highest form in one space on to the highest form in the other space.

To get an insight into how the overlap formula realizes the chiral determinant, one can look at the process formally in the continuum.

The ground state of  $a^\dagger \gamma_5 a$  is obtained by filling all the states of negative chirality, namely, states of the form

$$\begin{pmatrix} 0 \\ v_k \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

To get an expression for the ground state of  $a^\dagger H(m)a$ , we can look at  $H(m)$  in the continuum, namely,

$$H(m) = \begin{pmatrix} m & C \\ C^\dagger & -m \end{pmatrix}; \quad m < 0$$

and observe that the ground state is obtained by filling all the states of the form

$$\frac{1}{\sqrt{N_{kk}}} \begin{pmatrix} u_k \\ C^\dagger \frac{1}{m - \sqrt{CC^\dagger + m^2}} u_k \end{pmatrix}$$

$$CC^\dagger u_k = (\lambda_k^2 - m^2) u_k; \quad N_{kk'} = \langle u_k \left[ 1 + \frac{CC^\dagger}{(m - \sqrt{CC^\dagger + m^2})^2} \right] | u_{k'} \rangle$$

This indicates that  $\langle 0 - | 0+ \rangle$  is proportional to  $\det C^\dagger$  in the limit where  $|m|$  is taken to infinity.

## Phase of $|0+\rangle$

Clearly, this procedure does not fix the phase of  $|0+\rangle$  since it is only defined as an eigenvector of a Hamiltonian.

The details involved in the phase choice and possible gauge breaking is the subject of chiral gauge theories.

Motivated by quantum mechanics, we can insist that

$${}_0\langle 0+ | 0+\rangle_A$$

is **real and positive** for all gauge fields.

Under a gauge transformation,

$$\langle 0+ | \mathcal{G} | 0+\rangle_A = \langle 0+ | 0+\rangle_A$$

But

$\langle 0+ | 0+\rangle_A$  and  $\langle 0+ | \mathcal{G} | 0+\rangle_A$  need not be both real and positive.

$\Rightarrow$  *Anomalies*

## Consistent and Covariant anomalies

$$\partial_\alpha |0+\rangle = \left[ \partial_\alpha |0+\rangle \right]_\perp + |0+\rangle \langle 0+ | \partial_\alpha |0+\rangle$$

$\partial_\alpha$  denotes the derivative with respect to the gauge field  $A_\mu^a(x) \equiv \alpha$ .

$$\frac{\langle \langle 0- | \partial_\alpha |0+\rangle \rangle}{\langle 0- | 0+\rangle} = \partial_\alpha \ln \langle 0- | 0+\rangle = \text{consistent current}$$

One can explicitly show that ( Neuberger, Phys. Rev. D59 (1999)

085006 )

$$\frac{\langle \langle 0- | \left[ \partial_\alpha |0+\rangle \right]_\perp \rangle \rangle}{\langle 0- | 0+\rangle}$$

transforms covariantly under a gauge transformation. This is the covariant current.

$$\Delta j = \langle 0+ | \partial_\alpha |0+\rangle d\alpha$$

is the piece connects the covariant and consistent currents.

## Berry's connection and curvature

$$\Delta j = \langle 0+ | \partial_\alpha | 0+ \rangle d\alpha$$

is the Berry's connection and does depend on the phase choice.

But

$$\Omega = d\Delta j = \langle d0+ | \wedge d0+ \rangle$$

is the Berry curvature and is independent of the phase choice. This is defined in a massive theory and is expected to be a local functional of the gauge fields.

$$\Omega = 0 \Leftrightarrow \text{No anomalies}$$

An explicit calculation shows that

$$\Omega = -\frac{1}{8} \text{Tr} \left( \epsilon(H) \left[ \partial_\alpha \epsilon(H), \partial_\beta \epsilon(H) \right] \right) d\alpha d\beta$$



## Vector gauge theories

We want

$$\det CC^\dagger = |\langle 0 - |0+\rangle|^2$$

Phase choice does not matter

**But the determinant is realized as a product of two chiral factors and chiral symmetry is built in.**

### Numerical method

Diagonalize  $H_w$  on the lattice.

Form the many body state from the negative energy single particle states.

This is too time consuming.

Question: Can we get to the many body state directly?

Answer: Yes. Neuberger solved this problem in **Phys. Lett.**

**B 417 (1998) 141** and is now referred to as the overlap Dirac operator.

The massless overlap Dirac operator is derived from the overlap formalism as follows:

$$H_w U = U \lambda; \quad U = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}$$

$$\epsilon(H_w) \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} = \begin{pmatrix} \alpha & -\gamma \\ \beta & -\delta \end{pmatrix}$$

$$\det U = \frac{\det \alpha}{\det \delta^\dagger}$$

$$|\langle 0 - | 0 + \rangle|^2 = \det \delta \det \delta^\dagger$$

$$= \det \delta \det \alpha \det U^\dagger$$

$$= \det \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} U^\dagger$$

$$= \det \frac{1}{2} \left\{ \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} + \begin{pmatrix} \alpha & -\gamma \\ -\beta & \delta \end{pmatrix} \right\} U^\dagger$$

$$= \det \frac{1}{2} \left\{ \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha & -\gamma \\ \beta & -\delta \end{pmatrix} \right\} U^\dagger$$

$$= \det \frac{1}{2} \left[ U + \gamma_5 \epsilon(H_w) U \right] U^\dagger$$

$$= \det \frac{1}{2} \left[ 1 + \gamma_5 \epsilon(H_w) \right]$$