Conformal expansions:
A template for QCD predictions

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Outline:

- Introduction
- Skeleton expansion
- Conformal coefficients
- BLM scale setting
- Thrust as an example
- Summary and conclusions

Introduction

Perturbative expansion for a single-scale spacelike observable in massless QCD ($a = \alpha_s/\pi$),

$$R(Q^2) = R_{\text{QPM}}(Q^2) + R_0(Q^2)a(\mu^2) + R_1(Q^2,\mu^2)a^2(\mu^2) + R_2(Q^2,\mu^2,\beta_2)a^3(\mu^2) + \cdots$$

- Truncation at order $N \Rightarrow$ renormalisation scale (μ) and scheme ($\beta_2, \ldots$) dependence of order $\mathcal{O}\left(a^{N+1}(\mu^2)\right)$
- Asymptotic expansion: renormalons $\Rightarrow R_n \sim n!\beta_0^n$

Questions:

- Is there a preferred scheme
- How to choose μ ($\mu \sim Q$ to avoid large log's)
- Can renormalons be avoided

Effective charge $a_R(Q^2)$

$$\begin{aligned} R(Q^2) &= R_{\text{QPM}}(Q^2) + R_0(Q^2)a_R(Q^2) \\ a_R(Q^2) &= a(\mu^2) + r_1 a^2(\mu^2) + r_2 a^3(\mu^2) + \cdots \end{aligned}$$

Conformal limit: $da(\mu^2)/d\log\mu^2 = \beta = 0$.

- no scale-ambiguity
- no factorial growth due to renormalons

Example: Crewther relation between Adler D-function and polarized Bjorken sum-rule for DIS

$$(1 + a_D)(1 - a_{g_1}) \quad = \quad 1$$

where the effective charges a_D and a_{g_1} are defined by

$$D(Q^2) = Q^2 \frac{d\Pi(Q^2)}{dQ^2} \equiv N_C \sum_f e_f^2 \left[1 + a_D(Q^2)\right]$$

$$\int_0^1 \left[g_1^p(x, Q^2) - g_1^n(x, Q^2)\right] dx \equiv \frac{g_A}{6 g_V} \left[1 - a_{g_1}(Q^2)\right]$$

In general:

$$a_A = \sum_n c_n^{AB} a_B^n$$

- the coefficients c_n^{AB} depend on which coupling is used a expansion parameter
- in real life the coupling runs
  $a_A(Q) = \sum_n c_n^{AB} a_B^n(Q) + \beta_B(a_B(Q)) T_{AB}(a_B(Q))$

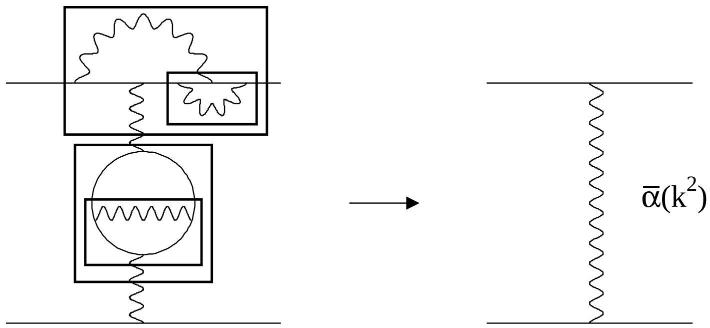
Commensurate scale relations

$$a_A(Q) = \sum_n c_n^{AB} a_B^n(Q^*)$$

Skeleton expansion

Skeleton graphs: fundamental vertices and propagators contain no substructure

Ex. QED



$\bar{\alpha}$ is the Gell-Mann Low coupling, resums the Dyson series of the 1PI photon self-energy Π

$$\bar{\alpha}(Q^2) = \frac{\alpha_0}{1 - \Pi(Q^2)}$$

$Z_1 = Z_2 \Rightarrow$ charge renormalisation given by photon propagator ren. (Z_3)

Radiative corrections to one-photon exchange skeleton can be written as

$$\int \bar{a}(k^2)\phi_0\left(\frac{k^2}{Q^2}\right)\frac{dk^2}{k^2}$$

where ϕ_0 is momentum distribution function normalised to 1. (Above example $\phi_0\left(\frac{k^2}{Q^2}\right) = \delta(k^2 - Q^2)$.)

Sum of one, two, three-photon exchange etc.

$$a_R(Q^2) = \int \bar{a}(k^2)\phi_0\left(\frac{k^2}{Q^2}\right)\frac{dk^2}{k^2} + \bar{c}_1\int \bar{a}(k_1^2)\bar{a}(k_2^2)\phi_1\left(\frac{k_1^2}{Q^2},\frac{k_2^2}{Q^2}\right)\frac{dk_1^2}{k_1^2}\frac{dk_2^2}{k_2^2} + \cdots$$

- skeleton coupling $\bar{a}(k^2)$ is gaugeinvariant
- each integral is renormalisation scheme and scale-invariant by itself
- $\bar{c}_i$ are the conformal coefficients in skeleton scheme ($\bar{a}(k^2) = \bar{a} =$ const. $\Rightarrow a_R = \bar{a} + \bar{c}_1\bar{a}^2 + \bar{c}_2\bar{a}^3 + \cdots$)
- ϕ_i are momentum distribution functions (normalised to 1)
- need diagrammatic construction to identify skeletons

QCD

More complicated due to gluon self-interactions: Assume simple ansatz similar to QED.

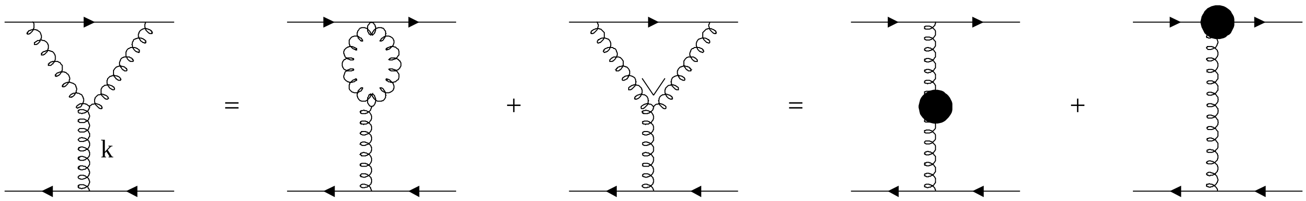
- one skeleton coupling
- only one skeleton at each order
- N_F-dependence of conformal coefficients can be identified (ex. light-by-light scattering)

Not clear that all-order skeleton expansion exists

Pinch technique

Realisation of skeleton expansion at one-loop order in QCD

Ex. three-gluon vertex



$\Rightarrow$ contributes to effective propagator and "external" vertex renormalisation

QED-like Ward identities:

$$Z_1^{(PT)} = Z_2^{(PT)} = 1 - \frac{1}{\varepsilon}\frac{C_F}{4}\bar{a}$$

$$Z_3^{(PT)} = 1 + \frac{1}{\varepsilon}\left(\frac{11}{12}C_A - \frac{1}{3}T_F N_F\right)\bar{a} = 1 + \frac{1}{\varepsilon}\beta_0 \bar{a}$$

$\Rightarrow$ all one-loop running coupling effects in effective gluon propagator

Relation to $\overline{\mathsf{MS}}$ scheme:

$$\bar{a}(Q^2) = a_{\overline{\mathsf{MS}}}(\mu^2) + \left[-\beta_0\left(\log\frac{Q^2}{\mu^2} - \frac{5}{3}\right) + 1\right] a^2_{\overline{\mathsf{MS}}}(\mu^2)$$

Conformal coefficients

Given the assumed skeleton expansion the first conformal coefficients can be obtained from the perturbative ones

$$
\begin{aligned}
a_R(Q^2) &= \int \bar{a}(k^2)\phi_0\left(\frac{k^2}{Q^2}\right)\frac{dk^2}{k^2} \\
&+ \bar{c}_1 \int \bar{a}(k_1^2)\bar{a}(k_2^2)\phi_1\left(\frac{k_1^2}{Q^2},\frac{k_2^2}{Q^2}\right)\frac{dk_1^2}{k_1^2}\frac{dk_2^2}{k_2^2} \\
&+\bar{c}_2 \int \bar{a}(k_1^2)\bar{a}(k_2^2)\bar{a}(k_3^2)\phi_2\left(\frac{k_1^2}{Q^2},\frac{k_2^2}{Q^2},\frac{k_3^2}{Q^2}\right)\frac{dk_1^2}{k_1^2}\frac{dk_2^2}{k_2^2}\frac{dk_3^2}{k_3^2} + \cdots
\end{aligned}
$$

Expand skeleton couplings under integration sign

$$
\bar{a}(k^2) = \bar{a}(Q^2) + \beta_0 \log(Q^2/k^2)\bar{a}^2(Q^2) + \left[\beta_1 \log(Q^2/k^2) + \beta_0^2 \log^2(Q^2/k^2)\right]\bar{a}^3(Q^2) + \cdots
$$

gives relation to perturbative coefficients

$$
\begin{aligned}
a_R(Q^2) &= \bar{a}(Q^2) + \left(\bar{c}_1 + \beta_0\phi_0^{(1)}\right)\bar{a}^2(Q^2) \\
&+ \left(\bar{c}_2 + \bar{c}_1\beta_0\phi_1^{(1)} + \beta_1\phi_0^{(1)} + \beta_0^2\phi_0^{(2)}\right)\bar{a}^3(Q^2) + \cdots \\
&= \bar{a}(Q^2) + r_1\bar{a}^2(Q^2) + r_2\bar{a}^3(Q^2) + \cdots
\end{aligned}
$$

$\phi_i^{(n)}$ are log-moments $\left(\phi_0^{(n)} = \int \log^n(Q^2/k^2)\phi_0\left(\frac{k^2}{Q^2}\right)\frac{dk^2}{k^2}\right)$

Use N_F dependence to identify contributions from different skeletons to the perturbative coefficients

$\Rightarrow$ the conformal coefficients and log-moments

Given assumed form of skeleton expansion there is a unique decomposition of the perturbative coefficients up to three-loop order

Possible breakdown of ansatz:

- more than one skeleton at each order
- N_F-dependent skeletons
- several skeleton couplings

General properties of ansatz may still be valid:

- running-coupling effects associated with different skeletons in a renormalisation group invariant way
- skeleton coefficients are the same as the conformal coefficients

In practice (usually) no problem at NLO

In conformal window ($8 < N_F < 16.5$) pQCD has infrared fixed-point ($k^2 \to 0$)

$$\frac{da_{\mathsf{FP}}(k^2)}{d\ln k^2} = -\beta_0 a^2_{\mathsf{FP}}(k^2) - \beta_1 a^3_{\mathsf{FP}}(k^2) + \cdots = 0$$

since $\beta_0 = \frac{11}{4} - \frac{1}{6}N_F > 0$ and $\beta_1 = \frac{51}{8} - \frac{19}{24}N_F < 0$

Expand a_{FP} in $a_0 = -\frac{\beta_0}{\beta_1|_{\beta_0=0}} = \frac{16}{107}\beta_0$

$$a_{\mathsf{FP}} = a_0 + v_1 a_0^2 + \cdots$$

v_i depends on higher order terms in β-function

Expand the coefficients r_i for an effective charge a_R in a_0 using polynomial N_F dependence

$$\begin{aligned} a_R(Q^2) &= a(Q^2) + (r_{1,0} + r_{1,1}a_0)a^2(Q^2) \\ &\quad + (r_{2,0} + r_{2,1}a_0 + r_{2,2}a_0^2)a^3(Q^2) + \cdots \end{aligned}$$

Take limit $Q^2 \to 0$ and insert $a_0 = a_{\mathsf{FP}} + u_1 a^2_{\mathsf{FP}} + \cdots$ gives

$$a_R^{\mathsf{FP}} = a_{\mathsf{FP}} + r_{1,0}a^2_{\mathsf{FP}} + (r_{2,0} + r_{1,1})a^3_{\mathsf{FP}} + \cdots$$

Comparison with skeleton expansion shows that the coefficients are the same, *i.e.* $r_{1,0} = \bar{c}_1$ and $r_{2,0} + r_{1,1} = \bar{c}_2$ etc.

Skeleton coefficients: Examples

Only known at NLO ($\bar{c}_1$) through pinch-technique

Adler D-function,

$$a_D = \bar{a} + \left(-\frac{1}{4}C_A - \frac{1}{8}C_F\right)\bar{a}^2 = \bar{a} - 0.917\bar{a}^2$$

The polarized Bjorken sum-rule for DIS,

$$a_{g_1} = \bar{a} + \left(-\frac{1}{4}C_A - \frac{7}{8}C_F\right)\bar{a}^2 = \bar{a} - 1.917\bar{a}^2$$

Static potential,

$$a_V = \bar{a} - C_A\bar{a}^2 = \bar{a} - 3\bar{a}^2$$

Also possible to calculate difference between $\bar{c}_2$ for pairs of observables

$$\begin{aligned}
\bar{c}_2^{g_1} - \bar{c}_2^D &= \frac{3}{8}C_F C_A + \frac{3}{4}{C_F}^2 = 2.833 \\
\bar{c}_2^V - \bar{c}_2^D &= \left[\frac{1}{4}\pi^2 + \frac{43}{24} - \frac{1}{64}\pi^4\right]{C_A}^2 \\
&- \frac{25}{16}C_F C_A + \frac{23}{32}{C_F}^2 = 19.66
\end{aligned}$$

Illustrates simplicity of conformal coefficients

Evaluating skeleton integrals

1. The leading skeleton integral can be evaluated using ϕ_0 calculated in the large β_0-approximation, renormalon ambiguity $\Rightarrow$ power-correction

2. Generalized effective charge method

3. BLM scale setting (and its generalisations)

$$
\begin{aligned}
a_R(Q^2) &= \int \bar{a}(k^2)\phi_0\left(\frac{k^2}{Q^2}\right)\frac{dk^2}{k^2} \\
&\quad + \bar{c}_1 \int \bar{a}(k_1^2)\bar{a}(k_2^2)\phi_1\left(\frac{k_1^2}{Q^2},\frac{k_2^2}{Q^2}\right)\frac{dk_1^2}{k_1^2}\frac{dk_2^2}{k_2^2} + \cdots \\
\text{(MVT)} &\equiv \bar{a}(l_0^2) + \bar{c}_1\bar{a}^2(l_1^2) + \cdots
\end{aligned}
$$

Requiring one-to-one correspondence $\Rightarrow$ unique scale setting procedure (l_0 depends on ϕ_0, l_1 on ϕ_1 etc.)

Expanding in $\bar{a}(Q^2)$ gives

$$
\ln\frac{Q^2}{l_0^2} = \underbrace{\phi_0^{(1)}}_{\text{mean}} + \underbrace{\left[\phi_0^{(2)} - \left(\phi_0^{(1)}\right)^2\right]}_{\text{variance}}\beta_0\bar{a}(l_0^2) + \cdots
$$

Provides systematic improvement of BLM-scale, $l^2_{0,\text{BLM}} = Q^2\exp\left(-\phi_0^{(1)}\right)$, but series is asymptotic.

(Skeleton integral calculated with approximation to ϕ.)

Similarly for multiple gluon exchange skeletons:

$$\ln \frac{Q^2}{l_1^2} = \frac{1}{2}\phi_1^{(1)} + \cdots$$

where

$$\phi_1^{(1)} = \int \left(\ln \frac{Q^2}{k_1^2} + \ln \frac{Q^2}{k_2^2} \right) \phi_1 \left(\frac{k_1^2}{Q^2}, \frac{k_2^2}{Q^2} \right) \frac{dk_1^2}{k_1^2} \frac{dk_2^2}{k_2^2}$$

Comparison with Commensurate Scale Relations:

- relates two arbitrary effective charges
- RGE-transitivity $\Rightarrow$ coefficients free of renormalons
- interpretation of scale not so clear (no unique scale setting procedure)

Example:

$$a_{g_1}(Q) = a_D(Q^*) - a_D^2(Q^*) + a_D^3(Q^*) + \cdots$$

where

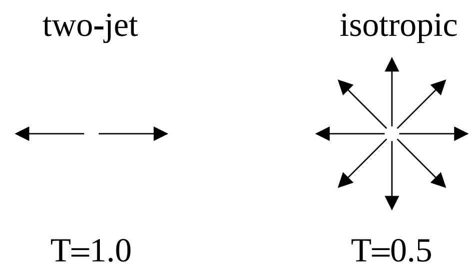
$$\begin{aligned} \ln \frac{Q^*}{Q} &= \frac{7}{4} - 2\zeta_3 + a_D(Q^*) \left[\left(\frac{11}{24} + \frac{28}{3}\zeta_3 - 8\zeta_3^2 \right) \beta_0 \right. \\ &\quad \left. + \left(\frac{13}{36} - \frac{1}{3}\zeta_3 \right) C_A + \left(-\frac{145}{144} - \frac{23}{3}\zeta_3 + 10\zeta_5 \right) C_F \right] \\ &= -0.654 + a_D(Q^*) \left[0.118\beta_0 + 0.0767 \right] \end{aligned}$$

Thrust as an example

Generalisation to time-like and multiscale observable

Thrust is an event shape variable in e^+e^--annihilation:

$$T = \max_{\vec{n}_T} \frac{\sum_i \vec{n}_T \cdot \vec{p}_i}{\sum_i |\vec{p}_i|}$$



One skeleton at LO, all N_F-dependence at NLO from running coupling effects

NLO conformal expansion with renormalon integral approximated by skeleton coupling at BLM-scale:

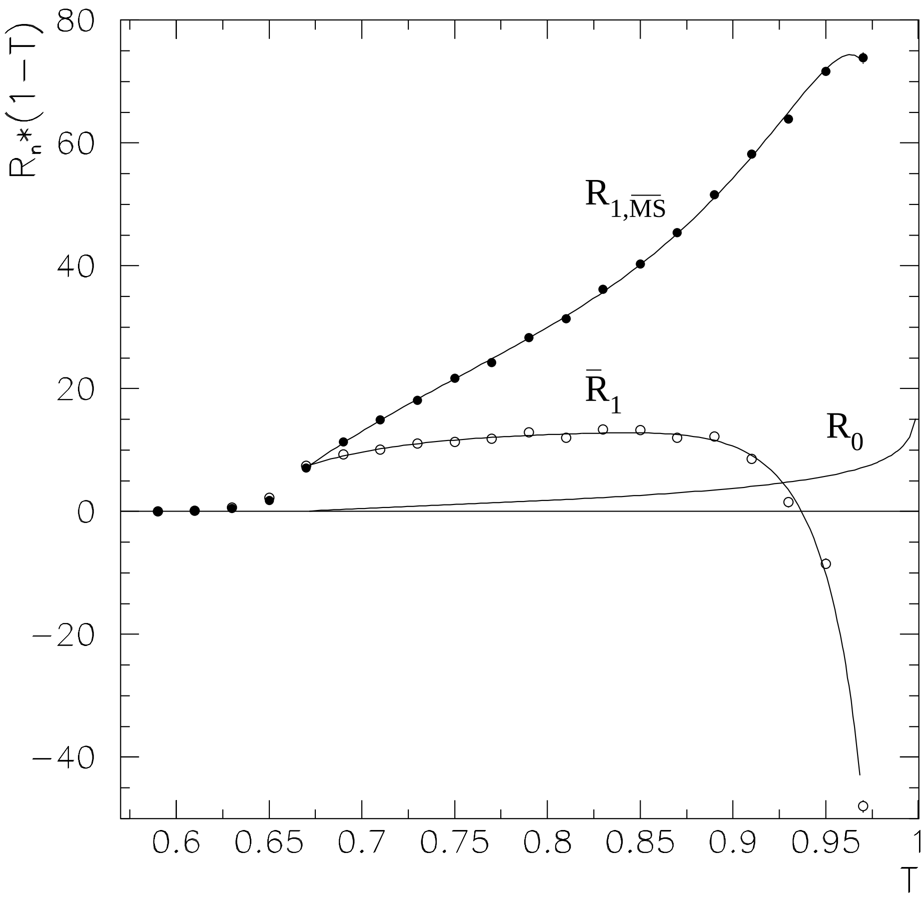
$$\frac{1}{\sigma_0}\frac{d\sigma^{PT}}{dT}(s,T) = \delta(1-T) + R_0(T)\bar{a}(l^2_{0,\mathrm{BLM}}) + \bar{R}_1(T)\bar{a}^2(l^2_{0,\mathrm{BLM}})$$

Use $l_1^2 = l^2_{0,\mathrm{BLM}}$ also for $\bar{R}_1(T)$-term

Compare with standard $\overline{\mathrm{MS}}$ expansion using $\mu^2 = s$

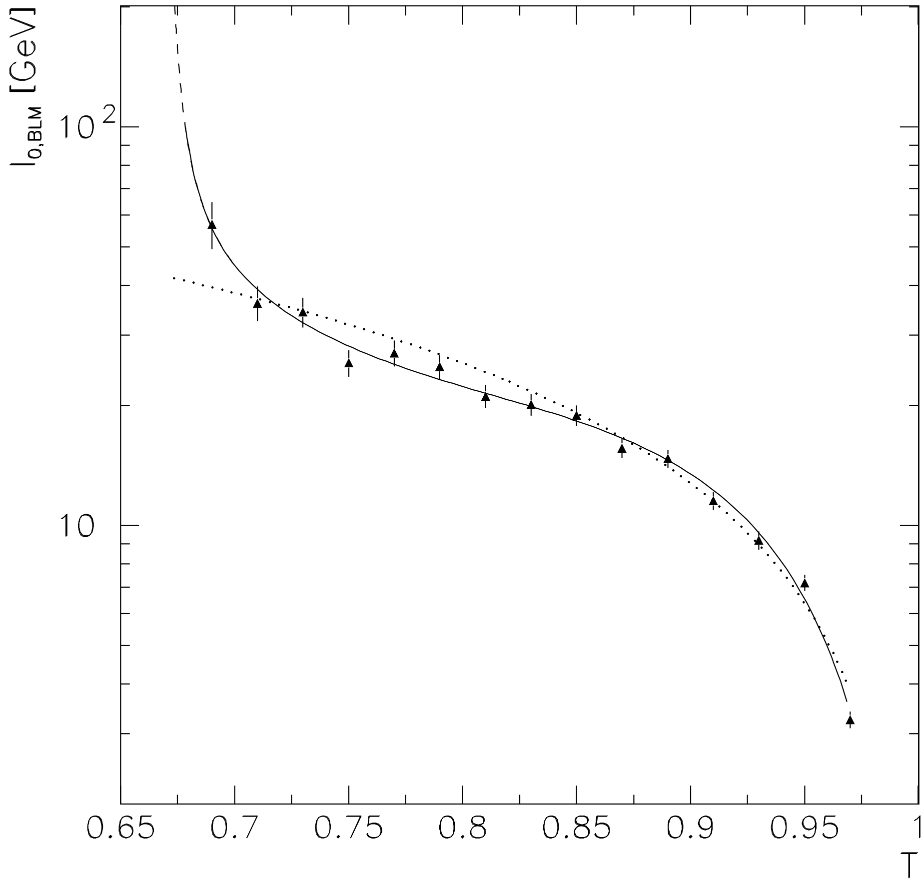
$$\frac{1}{\sigma_0}\frac{d\sigma^{\overline{\mathrm{MS}}}}{dT}(s,T) = \delta(1-T) + R_0(T)a_{\overline{\mathrm{MS}}}(s) + R_{1,\overline{\mathrm{MS}}}(\mu^2 = s, N_F, T)a^2_{\overline{\mathrm{MS}}}(s)$$

Conformal coefficient R_1 calculated numerically (using Beowulf) and compared with $R_{1,\overline{\text{MS}}}(\mu^2 = s, N_F = 5)$



- $R_0 \neq 0$ for $\frac{2}{3} < T < 1$
- $R_1 \neq 0$ for $\frac{1}{\sqrt{3}} < T < 1$
- $\bar{R}_1$ negative for $1 - T \ll 1$, need for resummation in Sudakov form-factor
- $\bar{R}_1 - R_{1,\overline{\text{MS}}} \to 0$ as $T \to 2/3$

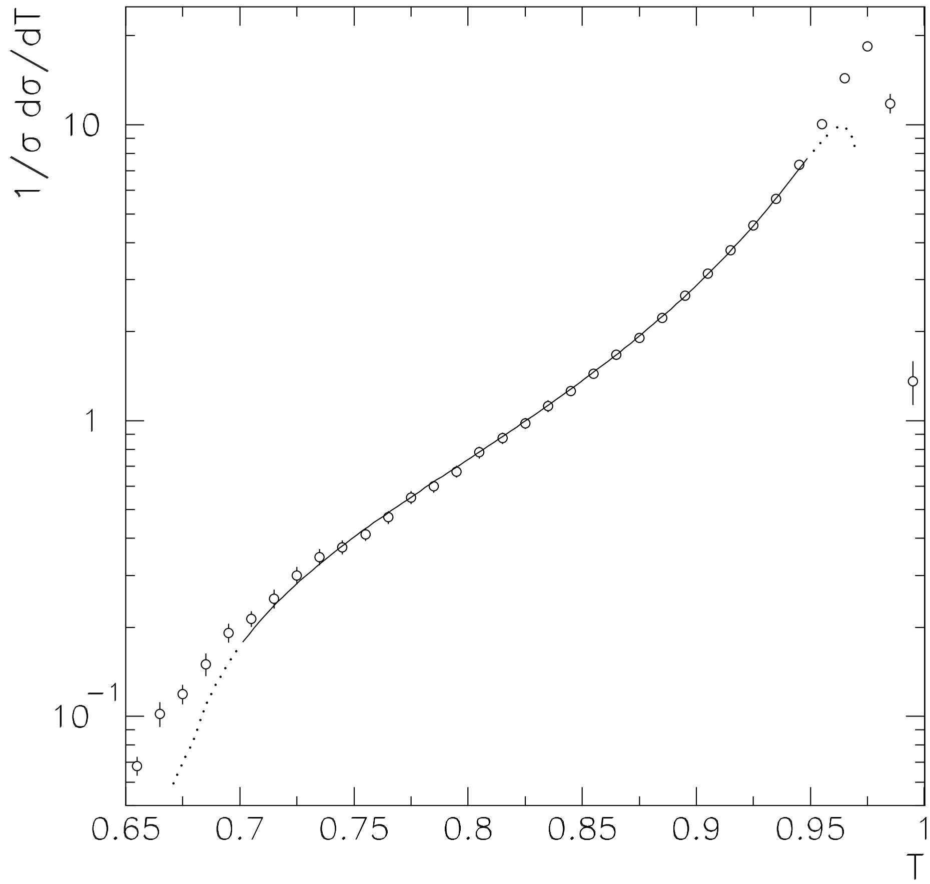
BLM scale $l_{0,\text{BLM}}(T)$ for $\sqrt{s} = M_Z$



- $l_{0,\text{BLM}} \simeq 1.4(1-T)\sqrt{s}$ (dotted line)

Fit to data from OPAL at $s = M_Z$ for $0.7 < T < 0.95$

$$\frac{1}{\sigma}\frac{d\sigma}{dT} = \frac{1}{\sigma_0}\frac{d\sigma^{PT}}{dT} + \frac{\Lambda}{(1-T)\sqrt{s}}$$



Result:

$\alpha_{\overline{\mathsf{MS}}}(M_Z^2)^*$	Λ [GeV]	$\frac{\chi^2}{d.o.f.}$
0.112 (0.115)	0.40	0.47
0.114 (0.117)	-	0.88

*Starting point for evolution $\bar{a}(e^{5/3}M_Z^2) = a_{\overline{\mathsf{MS}}}(M_Z^2) + a^2_{\overline{\mathsf{MS}}}(M_Z^2)$ $(\bar{a}(M_Z^2) = a_{\overline{\mathsf{MS}}}(M_Z^2) + (5/3\beta_0 + 1)a^2_{\overline{\mathsf{MS}}}(M_Z^2))$

cf. Standard $\overline{\mathsf{MS}}$: $\alpha_{\overline{\mathsf{MS}}}(M_Z^2) = 0.143$, $\chi^2/d.o.f. = 0.54$

cf. Average thrust in single dressed gluon approximation (Gardi and Grunberg) $\Rightarrow \alpha_{\overline{\mathsf{MS}}}(M_Z^2) = 0.110 \pm 0.002$

Summary and conclusions

Standard perurbative expansion:

- renormalisation scheme and scale ambiguities
- factorial growth of coefficients due to renormalons

The skeleton expansion:

- no renormalon growth of coefficients
- no scale (or scheme) ambiguity
- conformal coefficients the same as with infrared fixed-point
- renormalon ambiguity gives scaling of non-perturbative contributions
- BLM scale setting method – manifestation of skeleton expansion
- presently only known to NLO in QCD

Use of conformal coefficients can make large difference for phenomenology

Ex. Thrust at NLO:

- Skeleton expansion gives $\alpha_{\overline{\mathsf{MS}}}(M_Z^2) = 0.114$
- compared to standard $\overline{\mathsf{MS}}$ result $\alpha_{\overline{\mathsf{MS}}}(M_Z^2) = 0.143$