

“Strategy of Regions” —
Expansions of Feynman Diagrams
both in Euclidean and
Pseudo-Euclidean Regimes

V.A. Smirnov (NPI MSU Moscow)

Carmel, CA

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1. Limits of momenta and masses
2. Strategy of subgraphs and strategy of regions for limits typical for Euclidean space
3. Strategy of regions for limits typical for pseudo-Euclidean space
 - 3.1 Threshold expansion
 - 3.2 Sudakov limit
 - 3.1 Regge limit.
Expansion versus analytical evaluation
4. Present status of strategy of regions

1. A graph $\Gamma \implies$ Feynman integral

$$F_{\Gamma}(m_1, m_2, \dots, q_1, q_2, \dots)$$

It really depends on the masses m_i and kinematical invariants $s_{ij} = q_i \cdot q_j$

A limit (regime): a decomposition of parameters into small and large ones.

I. Limits typical for Euclidean space:

$$\{m_i\}, \{q_i\} \rightarrow \{m_i\}, \{q_i\}; \{M_i\}, \{Q_i\}$$

$$\text{with } m_i, |q_j| \ll M_{i'}, |Q_{j'}|$$

II. Limits typical for pseudo-Euclidean space:

$$\{m_i\}, \{s_{jj'}\} \implies \{m_i\}, \{s_{jj'}\}; \{M_i\}, \{S_{jj'}\}$$

$$\text{with } m_i, |s_{jj'}| \ll M_{i'}, |S_{jj'}|$$

(NB: combinations of $s_{jj'}$ can be used)

A limit \implies asymptotic expansion in powers and logs of ratios of small and large parameters.

E.g. $F_{\Gamma}(q^2, m^2)$ in the limit $m^2 \ll -q^2$ is expanded as

$$\sim (-q^2)^{\omega} \sum_{n=n_0}^{\infty} \sum_{j=0}^{2h} C_{nj} \left(\frac{-m^2}{q^2} \right)^n \ln^j \left(\frac{-m^2}{q^2} \right)$$

(h loop number, ω degree of divergence)

How to expand?

1. Take a given diagram in a given limit and expand it by some technique
2. Formulate prescriptions for a given limit, apply them to any diagram (e.g. with 100 loops)

2nd (global) solution \rightarrow

- *no analytical work*: just follow prescriptions and write down a result in terms of Feynman integrals (with integrands expanded in Taylor series in some parameters);
- evaluation of terms in the expansion without evaluating the full result;
- a natural requirement: individual terms of the expansion are homogeneous (modulo logs) in the expansion parameter;
- factorization of scales.

Two kinds of global prescriptions:

- * *Strategy of Subgraphs*
- * *Strategy of Regions*

2. For limits typical for Euclidean space: a solution is given by the strategy of subgraphs

[S.G. Gorishny, Nucl. Phys. **B319** (1989) 633;

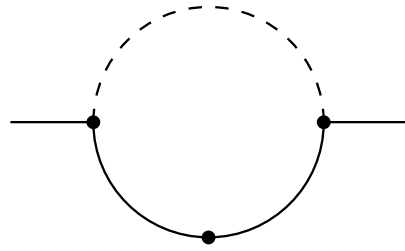
K.G. Chetyrkin, Teor. Mat. Fiz. **75** (1988) 26; *ibid.* **76** (1988) 207; V.A. Smirnov, Commun. Math. Phys. **134** (1990) 109 (analytical part of the proof)]

A very simple prescription

$$F_{\Gamma} \sim \sum_{\gamma} F_{\Gamma/\gamma} \circ \mathcal{T}_{\gamma} F_{\gamma}$$

- the sum runs in a certain class of subgraphs γ of Γ ; e.g. in the limit $m^2 \ll -q^2$, one can distribute the flow of q through *all* the lines of γ ;
- F_{γ} and $F_{\Gamma/\gamma}$ are the Feynman integrals resp. for γ and Γ/γ (the latter obtained from Γ by collapsing γ to a point);
- \mathcal{T}_{γ} expands the integrand of $F_{\Gamma/\gamma}$ in Taylor series in its small masses and small external momenta which are either the small external momenta of Γ , or loop momenta of the whole graph that are external for γ (they are *by definition* small);
- \circ insertion of the 2nd factor (polynomial) into $F_{\Gamma/\gamma}$ (like an insertion of a counterterm);
- all quantities are dimensionally regularized by $d = 4 - 2\varepsilon$ (An interplay between UV and IR divergences.)

An example



$$F_{\Gamma}(q^2, m^2; \varepsilon) = \int \frac{d^d k}{(k^2 - m^2)^2 (q - k)^2},$$

$k^2 - m^2 + i0$, etc.

Two subgraphs give non-zero contributions:

$\Gamma \rightarrow$ Taylor expansion of the integrand in m :

$$\begin{aligned} & \int \frac{d^d k}{(q - k)^2} \mathcal{T}_m \frac{1}{(k^2 - m^2)^2} \\ &= \int \frac{d^d k}{(k^2)^2 (q - k)^2} - 2m^2 \int \frac{d^d k}{(k^2)^3 (q - k)^2} + \dots \\ &= \frac{i\pi^{d/2}}{(-q^2)^{1+\varepsilon}} \frac{\Gamma(1 - \varepsilon)^2 \Gamma(\varepsilon)}{\Gamma(1 - 2\varepsilon)} \left[1 + 2\varepsilon \frac{m^2}{q^2} + \dots \right] \end{aligned}$$

$\gamma_1 = \{\text{upper line}\} \rightarrow$ Taylor expansion of its propagator in k :

$$\begin{aligned}
& \int \frac{d^d k}{(k^2 - m^2)^2} \mathcal{T}_k \frac{1}{(q - k)^2} \\
&= \frac{1}{q^2} \int \frac{d^d k}{(k^2 - m^2)^2} + \frac{1}{(q^2)^2} \int \frac{(2q \cdot k - k^2) d^d k}{(k^2 - m^2)^2} + \dots \\
&= \frac{i\pi^{d/2}}{q^2 (m^2)^\varepsilon} \Gamma(\varepsilon) \left[1 + \frac{\varepsilon}{1 + \varepsilon} \frac{m^2}{q^2} + \dots \right]
\end{aligned}$$

$\gamma = \{\text{two lower lines}\} \rightarrow$

$$\int \frac{d^d k}{k^2} \mathcal{T}_{k,m} \frac{1}{((q - k)^2 - m^2)^2} = \frac{1}{(q^2)^2} \int \frac{d^d k}{k^2} + \dots = 0$$

When $\varepsilon \rightarrow 0$, IR poles in the 1st contribution are canceled against UV poles in the second one, with the finite result

$$F_\Gamma(q^2, m^2; 0) \sim \frac{i\pi^2}{q^2} \left[\ln \left(\frac{-q^2}{m^2} \right) - \frac{m^2}{q^2} + \dots \right]$$

3. No simple generalizations of the strategy of subgraphs to typical Minkowskian regimes.

Exceptions:

The large momentum off-shell limit and one of the versions of the Sudakov limit

[V.A. Smirnov, Phys. Lett. **B394** (1997) 205; **B404** (1997) 101; A. Czarnecki and V.A. Smirnov, Phys. Lett. **B394** (1997) 211.]

The (standard) strategy of regions:

- Consider various regions of the loop momenta and expand, in every region, the integrand in a Taylor series with respect to the parameters that are considered small in the given region;
- pick up the leading asymptotic behaviour generated by every region.

It gives the leading power and (sub)leading logarithms. Cut-offs that specify the regions are not removed. The leading asymptotics are generated only by specific regions.

The generalized strategy of regions [M. Beneke and V.A. Smirnov, Nucl. Phys. **B522** (1998) 321]

- Consider various regions . . .
- Integrate the integrand expanded, in every region in its own way, over the whole integration domain in the loop momenta;
- Put to zero any integral without scale.

NB: For typically Euclidean limits, integrals without scale (tadpoles) are *automatically* put to zero. For general limits, this is an *ad hoc* prescription.

An experimental observation:

The strategy of regions gives asymptotic expansions for any diagram in any limit.

Comparison with existing explicit analytical results.

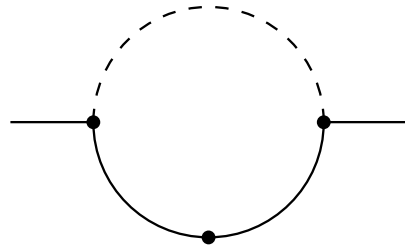
An indirect confirmation:

For limit typical for Euclidean space \rightarrow the same prescriptions as within the strategy of subgraphs:

Take any loop momentum to be either

$$\begin{aligned} \textit{large} : & \quad k \sim q , \quad \text{or} \\ \textit{small} : & \quad k \sim m . \end{aligned}$$

The same example:



$$F_{\Gamma}(q^2, m^2; \varepsilon) = \int \frac{d^d k}{(k^2 - m^2)^2 (q - k)^2}$$

$$k \text{ large: } \rightarrow \mathcal{T}_m \frac{1}{(k^2 - m^2)^2} \leftrightarrow \Gamma$$

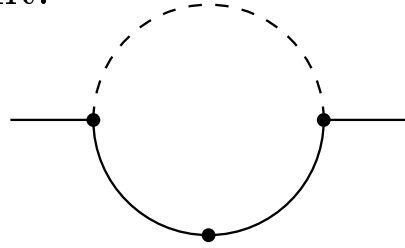
$$k \text{ small: } \rightarrow \mathcal{T}_k \frac{1}{(q - k)^2} \leftrightarrow \gamma_1$$

Lines with large momenta (loop and external) \leftrightarrow subgraph γ in the sum present within the strategy of subgraphs

3.1 Threshold expansion

[M. Beneke and V.A. Smirnov, NPB (1998)]

The same diagram in another limit:



$$\begin{aligned}
 F_{\Gamma}(q^2, m^2; \varepsilon) &= \int \frac{d^d k}{k^2((q-k)^2 - m^2)^2} \\
 &= \int \frac{d^d k}{k^2(k^2 - 2q \cdot k - y)^2}
 \end{aligned}$$

$$(q^2, m^2) \rightarrow (q^2, y): y = m^2 - q^2 \rightarrow 0.$$

Look for relevant regions. Large \equiv hard (h): $k \sim q \rightarrow$

$$\begin{aligned}
 \int \frac{d^d k}{k^2} \mathcal{T}_y \frac{1}{(k^2 - 2q \cdot k - y)^2} &= \int \frac{d^d k}{k^2(k^2 - 2q \cdot k)^2} + \dots \\
 &= \frac{i\pi^{d/2}}{(q^2)^{1+\varepsilon}} \frac{\Gamma(1+\varepsilon)}{2\varepsilon} + \dots
 \end{aligned}$$

What else? Small \equiv soft (s): $k \sim \sqrt{y} \rightarrow$

$$\int \frac{d^d k}{k^2(-2q \cdot k)^2} + \dots = 0$$

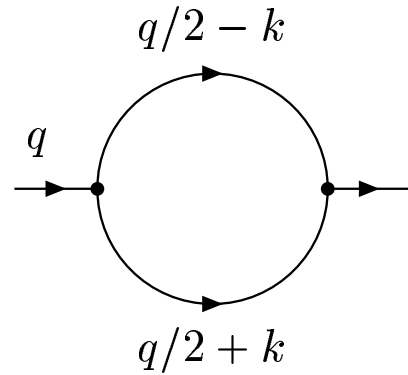
Try *ultrasoft* (us): $k \sim y/\sqrt{q^2} \rightarrow$

$$\begin{aligned} \int \frac{d^d k}{k^2} \mathcal{T}_{k^2} \frac{1}{(k^2 - 2q \cdot k - y)^2} &= \int \frac{d^d k}{k^2 (-2q \cdot k - y)^2} + \dots \\ &= -i\pi^{d/2} \frac{\Gamma(1 - \varepsilon)\Gamma(2\varepsilon)}{(q^2)^{1-\varepsilon} y^{2\varepsilon}} \end{aligned}$$

(h)+(us) when $\varepsilon \rightarrow 0$:

$$\frac{i\pi^{d/2}}{q^2} \left[\ln \frac{y}{q^2} - \frac{y}{q^2} + \dots \right]$$

An example with two massive lines in the cut



$$\begin{aligned}
 F_{\Gamma}(q^2, m^2; \varepsilon) &= \int \frac{d^d k}{(k^2 - m^2)((q - k)^2 - m^2)} \\
 &= \int \frac{d^d k}{(k^2 + q \cdot k - y)(k^2 - q \cdot k - y)}
 \end{aligned}$$

$$(q^2, m^2) \rightarrow (q^2, y): y = m^2 - q^2/4 \rightarrow 0.$$

$$\text{Choose } q = \{q_0, \vec{0}\}$$

Look for relevant regions. (h): $k \sim q \rightarrow$

$$\begin{aligned}
 &\int d^d k \mathcal{T}_y \frac{1}{(k^2 + q_0 k_0 - y)(k^2 - q_0 k_0 - y)} + \dots \\
 &= \int d^d k \frac{1}{(k^2 + q_0 k_0)(k^2 - q_0 k_0)} + \dots \\
 &= i\pi^{d/2} \left(\frac{4}{q^2}\right)^\varepsilon \frac{\Gamma(\varepsilon)}{1 - 2\varepsilon} + \dots
 \end{aligned}$$

What else? (s)→

$$-\frac{1}{q^2} \int \frac{d^d k}{k_0^2} + \dots = 0$$

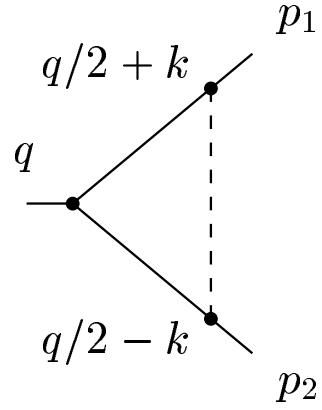
(us)→

$$-\frac{1}{q^2} \int \frac{dk_0 d^d \vec{k}}{(q_0 k_0 - y + i0)(q_0 k_0 + y - i0)} + \dots = 0$$

Try *potential* (p): $k_0 \sim y/q_0$, $\vec{k} \sim \sqrt{y} \rightarrow$ Taylor expansion in k_0^2 :

$$\begin{aligned} \int \frac{dk_0 d^{d-1} \vec{k}}{(\vec{k}^2 - q_0 k_0 + y - i0)(\vec{k}^2 + q_0 k_0 + y - i0)} + \dots \\ = i\pi^{d/2} \Gamma(\varepsilon - 1/2) \sqrt{\frac{\pi y}{q^2}} y^{-\varepsilon} \end{aligned}$$

(h)+(p) → the whole result



$$\int \frac{d^d k}{(k^2 + q \cdot k - y)(k^2 - q \cdot k - y)(k - p)^2}$$

$$q = p_1 + p_2, \quad p = (p_1 - p_2)/2, \quad p_1^2 = p_2^2 = m^2, \\ y = m^2 - q^2/4 \rightarrow 0.$$

$$\text{Choose } q = \{q_0, \vec{0}\}$$

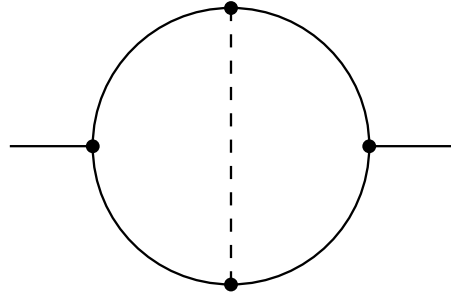
Look for relevant regions. (h): $k \sim q \rightarrow$

$$\int d^d k \frac{1}{(k^2 + q_0 k_0)(k^2 - q_0 k_0)(k - p)^2} \\ = -i\pi^{d/2} \left(\frac{4}{q^2}\right)^{1+\varepsilon} \frac{\Gamma(\varepsilon)}{2(1+2\varepsilon)} + \dots$$

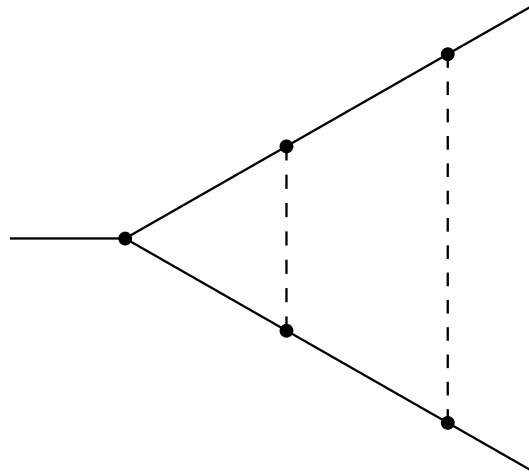
(s) $\rightarrow 0$; (us) $\rightarrow 0$; (p) \rightarrow

$$\int \frac{dk_0 d^{d-1} \vec{k}}{(\vec{k}^2 - q_0 k_0 + y - i0)(\vec{k}^2 + q_0 k_0 + y - i0)(-(\vec{k} - \vec{p})^2)} \\ = i\pi^{d/2} \frac{y^{-\varepsilon}}{\sqrt{q^2 y}} \frac{\sqrt{\pi} \Gamma(\varepsilon + 1/2)}{2\varepsilon}$$

(h)+(p) \rightarrow the whole result



(h-h), (h-p)=(p-h), (p-p), (p-us)



(h-h), (h-p), (p-h), (p-p), (p-s), (p-us)

Transition to the operator level:

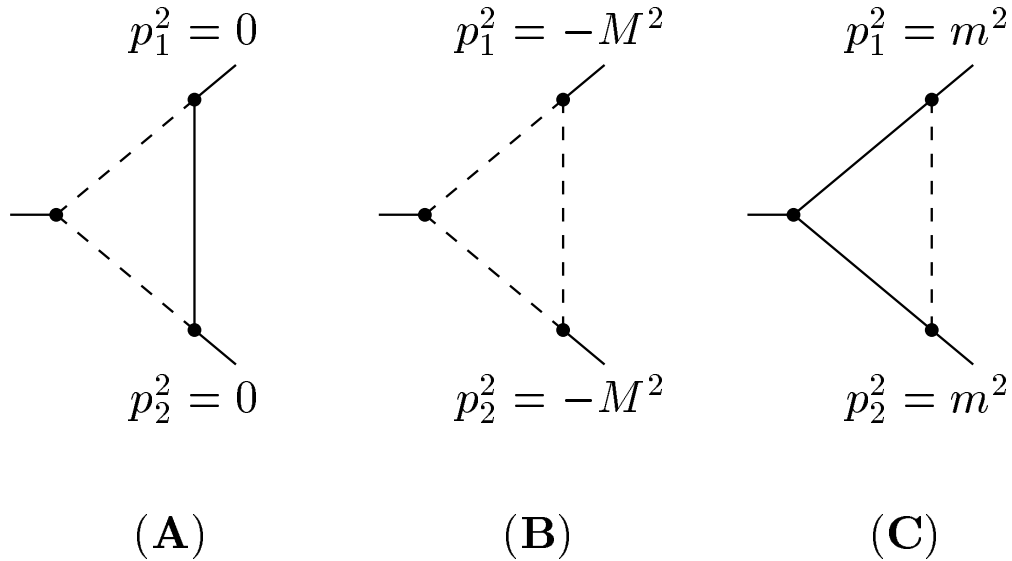
One zero (small) and one-non zero mass in the threshold \rightarrow HQET

Two non-zero masses in the threshold \rightarrow NRQCD
 \rightarrow pNRQCD

Applications of the threshold expansion:

e.g. $e^+e^- \rightarrow t\bar{t}$, see a review [A.Hoang, hep-ph/0001286]

3.2 Sudakov limit: $m^2 \ll Q^2 \equiv -s = -(p_1 - p_2)^2$
 or $M^2 \ll -s$



”Standard” strategy of regions \rightarrow

Summing up (sub)leading logarithms (at the leading power), evolution equations

[V.V. Sudakov, Zh. Eksp. Teor. Fiz. 30 (1956) 87;
 R. Jackiw, Ann. Phys. 48 (1968) 292; 51 (1969) 575;
 J.M. Cornwall and G. Tiktopoulos, Phys. Rev. Lett. 35
 (1975) 338; D13 (1976) 3370; J. Frenkel and J.C. Taylor,
 Nucl. Phys. B116 (1976) 185; J.C. Collins, Phys.Rev.
 D22 (1980) 1478; in *Perturbative QCD*, ed. A.H. Mueller,
 1989, p. 573. A. Sen, Phys. Rev. D24 (1981) 3281; D28
 (1983) 860; G. Korchemsky, Phys. Lett. B217 (1989) 330;
 B220 (1989) 629]

A: choose $p_{1,2} = (Q/2, 0, 0, \mp Q/2)$

$$\int \frac{d^d k}{(k^2 - 2p_1 \cdot k)(k^2 - 2p_2 \cdot k)(k^2 - m^2)}$$

$$= \int \frac{dk_+ dk_- d^{d-2} \underline{k}}{(k_+ k_- - \underline{k}^2 - Qk_+)(k_+ k_- - \underline{k}^2 - Qk_-)(k_+ k_- - \underline{k}^2 - m^2)}$$

where $k_{\pm} = k_0 \pm k_3$, $\underline{k} = (k_1, k_2)$,
with $2p_{1,2} \cdot k = Qk_{\pm}$.

Look for relevant regions.

(h): $k \sim q \rightarrow$ Taylor expansion in m^2

$$\int \frac{d^d k}{(k^2 - 2p_1 \cdot k)(k^2 - 2p_2 \cdot k)k^2} + \dots$$

$$= -i\pi^{d/2} \frac{1}{(Q^2)^{1+\varepsilon}} \frac{\Gamma(1+\varepsilon)\Gamma(-\varepsilon)^2}{\Gamma(1-2\varepsilon)} + \dots$$

(s): $k \sim m \rightarrow$

$$\int \frac{d^d k}{(-2p_1 \cdot k)(-2p_2 \cdot k)k^2} + \dots = 0$$

(us): $k \sim m^2/Q \rightarrow$

$$\int \frac{d^d k}{(-2p_1 \cdot k)(-2p_2 \cdot k)(-m^2)} + \dots = 0$$

Collinear regions were introduced in the
"standard strategy of regions":

[G. Sterman Phys. Rev. D17 (1978) 2773; S.Libby and
G. Sterman Phys. Rev. D18 (1978) 3252; A.H. Mueller,
Phys. Rep. 73 (1981) 35]

1-collinear (1c): $k_+ \sim m^2/Q, k_- \sim Q, \underline{k} \sim m,$

2-collinear (2c): $k_+ \sim Q, k_- \sim m^2/Q, \underline{k} \sim m.$

(1c): \rightarrow expansion of propagator 2 in k^2

$$\int \frac{d^d k}{(k^2 - 2p_1 \cdot k)(-2p_2 \cdot k)(k^2 - m^2)} + \dots$$

Introduce an auxiliary analytic regularization,
calculate (1c) and (2c) and switch it off in the sum
(1c)+(2c) \rightarrow

$$-i\pi^{d/2} \frac{\Gamma(\varepsilon)}{Q^2(m^2)^\varepsilon} \\ \times \left[\ln(Q^2/m^2) + \psi(\varepsilon) - \gamma_E - 2\psi(1 - \varepsilon) \right] + \dots$$

(h)+(1c)+(2c) at $\varepsilon \rightarrow 0$: IR/collinear poles in (h) and UV/collinear poles in (c) are canceled, with the finite result

$$\frac{i\pi^2}{Q^2} \left[\text{Li}_2(x) - \frac{1}{2} \ln^2 x + \ln x \ln(1-x) - \frac{\pi^2}{3} \right],$$

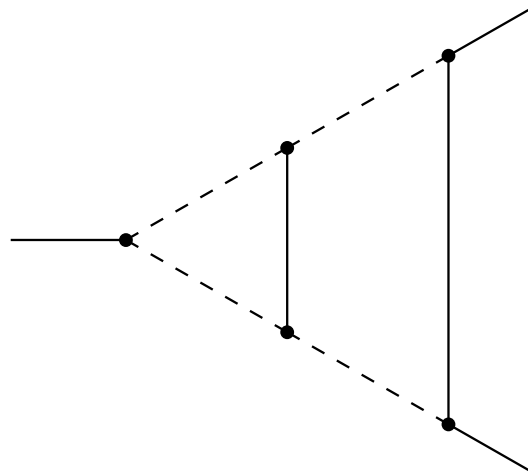
where $x = m^2/Q^2$.

B: (h), (1c), (2c), (us).

C: (h), (1c), (2c).

Two loops

A:



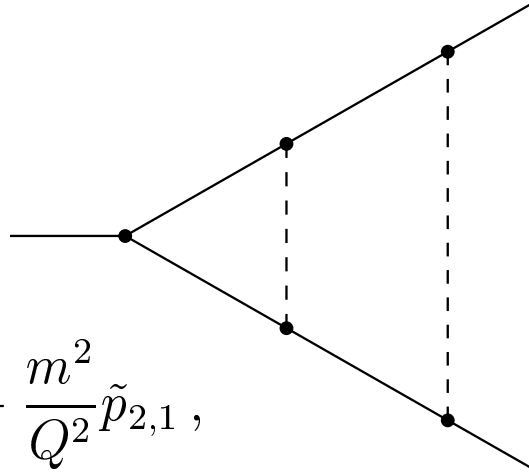
(h-h), (1c-h)+(2c-h), (1c-1c)+(2c-2c), (h-s)

B:

(h-h), (1c-h)=(2c-h), (1c-1c)=(2c-2c),

(us-h), (us-1c), (us-2c), (us-us)

C:



Choose

$$p_{1,2} = \tilde{p}_{1,2} + \frac{m^2}{Q^2} \tilde{p}_{2,1},$$

with $\tilde{p}_{1,2} = (Q/2, 0, 0, \mp Q/2)$

(h-h), (1c-h)=(2c-h), (1c-1c)=(2c-2c), what else?

Ultrascollinear:

(1uc): $k_+ \sim m^4/Q^3$, $k_- \sim m^2/Q$, $\underline{k} \sim m^3/Q^2$,

(2uc): $k_+ \sim m^2/Q$, $k_- \sim m^4/Q^3$, $\underline{k} \sim m^3/Q^2$.

Add (1uc-2c)+(1c-2uc).

[V.A. Smirnov, Phys. Lett. **B465** (1999) 226]

J.H. Kühn, A.A. Penin and V.A. Smirnov, hep-ph/9912503

Strategy of regions \rightarrow

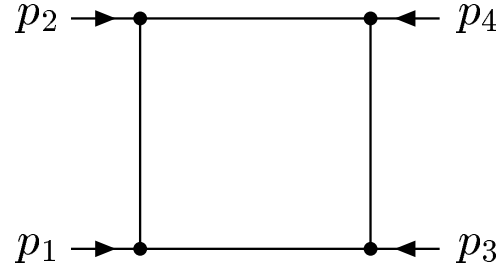
simple identification of regions contributing to functions that enter evolution equations (Abelian form factor and four-fermion amplitude in the $SU(N)$ gauge theory in the Sudakov limit up to the next-to-leading logarithmic approximation).

3.3 Regge limit:

$$|t| \ll |s|;$$

$$s = (p_1 + p_2)^2,$$

$$t = (p_1 + p_3)^2$$



$$\int \frac{d^d k}{(k^2 + 2p_1 \cdot k)(k^2 - 2p_2 \cdot k)k^2(k + p_1 + p_3)^2}$$

Let $s = -Q^2$, $t = -T$, $p_{1,2} = (\mp Q/2, 0, 0, Q/2)$,
 $p_1 + p_3 = (T/Q, 0, \sqrt{T + T^2/Q^2}, 0)$.

$$(1(2)c): \quad k_{\pm} \sim T/Q, \quad k_{\mp} \sim Q, \quad \underline{k} \sim \sqrt{T}$$

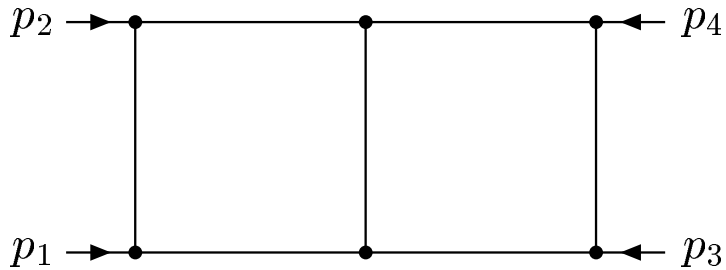
$$(1c+2c): \rightarrow$$

$$i\pi^{d/2} \frac{\Gamma(-\varepsilon)^2 \Gamma(1 + \varepsilon)}{\Gamma(-2\varepsilon) s (-t)^{1+\varepsilon}} \\ \times [\ln(t/s) + \psi(-\varepsilon) - 2\psi(1 + \varepsilon) + \gamma_E]$$

(h+1c+2c): \rightarrow only the LO (c) contribution survives at $\varepsilon \rightarrow 0$ and gives

$$\frac{i\pi^{d/2} e^{-\gamma_E \varepsilon}}{st} \left[\frac{4}{\varepsilon^2} - (\ln(-s) + \ln(-t)) \frac{2}{\varepsilon} \right. \\ \left. + 2 \ln(-s) \ln(-t) - \frac{4\pi^2}{3} \right]$$

On-shell massless double box, $p_i^2 = 0$



Evaluation of the double boxes: remember the talk by T. Gehrmann

$$\int \int \frac{d^d k d^d l}{(l^2 + 2p_1 \cdot l)(l^2 - 2p_2 \cdot l)(k^2 + 2p_1 \cdot k)(k^2 - 2p_2 \cdot k)} \times \frac{1}{k^2(k-l)^2(l+r)^2} \equiv \frac{(i\pi^{d/2} e^{-\gamma_E \varepsilon})^2}{(-s)^{2+2\varepsilon} (-t)} K(t/s; \varepsilon)$$

(h-h) starts from the NLO, t^0

(1c-1c)+(2c-2c) starts from LO, t^{-1}

LO+NLO [V.A. Smirnov and O.L. Veretin, Nucl. Phys. **B566** (2000) 469] \rightarrow

$$\begin{aligned} K(x, \varepsilon) = & -\frac{4}{\varepsilon^4} + \frac{5 \ln x}{\varepsilon^3} - \left(2 \ln^2 x - \frac{5}{2} \pi^2 \right) \frac{1}{\varepsilon^2} \\ & - \left(\frac{2}{3} \ln^3 x + \frac{11}{2} \pi^2 \ln x - \frac{65}{3} \zeta(3) \right) \frac{1}{\varepsilon} \\ & + \frac{4}{3} \ln^4 x + 6 \pi^2 \ln^2 x - \frac{88}{3} \zeta(3) \ln x + \frac{29}{30} \pi^4 \\ & + 2x \left[\frac{1}{\varepsilon} (\ln^2 x - 2 \ln x + \pi^2 + 2) - \frac{1}{3} (4 \ln^3 x + 3 \ln^2 x \right. \\ & \left. + (5\pi^2 - 36) \ln x + 2(33 + 5\pi^2 - 3\zeta(3))) \right] + O(x^2 \ln^3 x) \end{aligned}$$

Agreement with the full result

[V.A. Smirnov, Phys. Lett. **B460** (1999) 397]

$$\begin{aligned} K(x, \varepsilon) = & -\frac{4}{\varepsilon^4} + \frac{5 \ln x}{\varepsilon^3} - \left(2 \ln^2 x - \frac{5}{2} \pi^2\right) \frac{1}{\varepsilon^2} \\ & - \left(\frac{2}{3} \ln^3 x + \frac{11}{2} \pi^2 \ln x - \frac{65}{3} \zeta(3)\right) \frac{1}{\varepsilon} \\ & + \frac{4}{3} \ln^4 x + 6 \pi^2 \ln^2 x - \frac{88}{3} \zeta(3) \ln x + \frac{29}{30} \pi^4 \\ & - \left[2 \text{Li}_3(-x) - 2 \ln x \text{Li}_2(-x) - (\ln^2 x + \pi^2) \ln(1+x)\right] \frac{2}{\varepsilon} \\ & - 4 (S_{2,2}(-x) - \ln x S_{1,2}(-x)) + 44 \text{Li}_4(-x) \\ & - 4 (\ln(1+x) + 6 \ln x) \text{Li}_3(-x) \\ & + 2 \left(\ln^2 x + 2 \ln x \ln(1+x) + \frac{10}{3} \pi^2\right) \text{Li}_2(-x) \\ & + (\ln^2 x + \pi^2) \ln^2(1+x) \\ & - \frac{2}{3} (4 \ln^3 x + 5 \pi^2 \ln x - 6 \zeta(3)) \ln(1+x) \end{aligned}$$

- A curious example: it is easier to calculate the full result rather than expand
- Expansions can provide a crucial check of analytical results
- To calculate parameters of the reggeon, it is necessary to take just the collinear contributions rather than the full result

[A.A. Penin and V.A. Smirnov, unpublished]

4. Present status of the strategy of regions

No mathematical proofs, similar to the strategy of subgraphs (applied only to the limits typical for Euclidean space)

The word "region" is understood in the physical sense, one does not bother about "the decomposition of unity".

What to do when studying a new limit?

- look for regions, typical for the limit (probably, they are similar to regions connected with known limits);
- test one-loop examples by comparing with explicit results;
- check poles in ε ; if it is not satisfied look for missing regions;
- check expansion numerically;
- use the strategy of regions formulated in α -parameters [V.A. Smirnov, Phys. Lett. B465 (1999) 226], e.g. to avoid double counting;
- stay optimistic because, up to now, there were no counterexamples!