Progress on Two-Loop Non-Propagator Integrals

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RADCOR/2000
Carmel / California
14.09.2000
Introduction

Experimental precision for many $2 \rightarrow 2$ and $1 \rightarrow 3$ reactions has reached a level that demands theoretical predictions at next-to-next-to-leading order.

Examples:

- Bhabha-Scattering: $e^+e^- \rightarrow e^+e^-$
- Three Jet observables in $e^+e^-$
- DIS (2+1)-Jet production
- Hadron-Hadron 2-Jet and $V + 1$-Jet production
Calculation of Jet Observables

General Structure:

\( n \) jets, \( m \)-th order in perturbation theory

\( n \) partons, \( m \) loop

\[ \vdots \]

\( n + m - 1 \) partons, 1 loop

\( n + m \) partons, tree

- Jet algorithm acts differently on different partonic final states
  \[ \rightarrow \text{Optical theorem can not be applied} \]

- Divergencies from real and virtual contributions must be extracted before application of jet algorithm
Calculation of Jet Observables

Techniques for combining real and virtual contributions:

- **Phase space slicing**  
  G. Kramer et al.; W. Giele, N. Glover

- **Subtraction**  
  K. Ellis, D. Ross, A. Terrano; S. Catani, M. Seymour

- **Hybrid Subtraction**  
  N. Glover, M. Sutton

All techniques require the analytic calculation of the amplitudes for all subprocesses (or at least of their divergent parts).

Major missing ingredient for three-jet type observables in $e^+e^-$ at NNLO:

Two-loop four-point integrals with massless propagators and one off-shell leg
Generic structure of scalar two-loop integrals:

\[ I_{t,r,s}(p_1, \ldots, p_n) = \int \frac{d^d k}{(2\pi)^d} \frac{d^d l}{(2\pi)^d} \frac{1}{D_1^{m_1} \ldots D_t^{m_t} S_1^{m_1} \ldots S_q^{m_q}} \]

- \( D_i \) : massless scalar propagators
- \( S_i \) : scalar products involving loop momenta
- \( t \) : number of different propagators
- \( r = \sum_i m_i \) : dimension of denominator
- \( s = \sum_i n_i \) : dimension of numerator

Topology of Feynman graph defined by specifying the set of different propagators

\( \{D_1, \ldots, D_t\} \)
Reduction of Scalar Two-Loop Four-Point Functions

Identities:

- **Integration-by-parts (IBP)**
  
  \[ \int \frac{d^dk}{(2\pi)^d} \frac{d^dl}{(2\pi)^d} \frac{\partial}{\partial a^\mu} \left[ b^\mu f(k, l, p_i) \right] = 0 \]

  with: \( a^\mu = k^\mu, l^\mu \) and \( b^\mu = k^\mu, l^\mu, p_i^\mu \)

- **Lorentz Invariance (LI)**

  \[ \int \frac{d^dk}{(2\pi)^d} \frac{d^dl}{(2\pi)^d} \delta\varepsilon_{\nu}^{\mu} \left( \sum_i p_i^\nu \frac{\partial}{\partial p_i^\mu} \right) f(k, l, p_i) = 0 \]

For each two-loop four-point integral, one has 10 IBP and 3 LI identities.
Reduction of Scalar Two-Loop Four-Point Functions

The IBP and LI identities for $I_{t,r,s}$ relate:

$I_{t,r,s}$ : the integral itself

$I_{t-1,r,s}$ : simpler topology

$I_{t,r+1,s}, I_{t,r+1,s+1}$ : same topology, more complicated than $I_{t,r,s}$

$I_{t,r-1,s}, I_{t,r-1,s-1}$ : same topology, simpler than $I_{t,r,s}$

In numbers:

$t = 7$

<table>
<thead>
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<th>$r$</th>
<th>different $I_{t,r,s}$</th>
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# equations grows faster than # unknowns
→ Reduction possible (using MAPLE and FORM)
Example of a reducible integral:
Genuine one-scale master integrals:

R. Gonsalves; G. Kramer, B. Lampe

\[ p_{12} = A_3 \left(-p_{12}^2\right)^{d-3} \]

\[ p_{12} = A_{2, LO} \left(-p_{12}^2\right)^{d-4} \]

\[ p_{12} = A_4 \left(-p_{12}^2\right)^{d-4} \]

\[ p_{12} = A_6 \left(-p_{12}^2\right)^{d-6} \]
Calculation of Master Integrals

E. Remiddi

Multi-scale master integrals fulfil
inhomogeneous differential equations in their
external invariants.

For example:

$$s_{123} \frac{\partial}{\partial s_{123}} =$$

$$+ \frac{d - 4}{2} \frac{2s_{123} - s_{12}}{s_{123} - s_{12}}$$

$$- \frac{3d - 8}{2} \frac{1}{s_{123} - s_{12}}$$

$$s_{12} \frac{\partial}{\partial s_{12}} =$$

$$- \frac{d - 4}{2} \frac{s_{12}}{s_{123} - s_{12}}$$

$$+ \frac{3d - 8}{2} \frac{1}{s_{123} - s_{12}}$$
Two-loop four-point functions with one off-shell leg depend on three invariants: $s_{12}, s_{13}, s_{23}$

Computation from differential equations:

- express differential equations in $s_{123}$ (trivial homogeneous rescaling relation) and $y = s_{13}/s_{123}, z = s_{23}/s_{123}$ (inhomogeneous equations)

- solve differential equations with product ansatz
  \[ \mathcal{R}(y, z; s_{123}, \epsilon)\mathcal{H}(y, z; \epsilon) \]

- prefactor $\mathcal{R}$: rational function, can be determined from homogeneous part of equations in $y$ and $z$

- Laurent-series $\mathcal{H}$: expansion in $\epsilon$, with coefficients containing two-dimensional harmonic polylogarithms $H(\tilde{m}(z); y)$
Harmonic Polylogarithms (HPL) $H(m; x)$ are an extension of the Nielsen polylogarithms $Li_i(x)$ and $S_{i,j}(x)$. They have the following properties:

- HPL are linear independent
- HPL fulfill a product algebra:
  \[ H(\bar{a}; x)H(\bar{b}; x) = \sum H(\bar{a} \oplus \bar{b}; x) \]
- HPL form a closed set under class of integrations
  \[ \int_0^x dx \left( \frac{1}{x}, \frac{1}{1-x}, \frac{1}{1+x} \right) H(\bar{b}; x) \]
Harmonic Polylogarithms

Extension to Two-dimensional Harmonic Polylogarithms (2dHPL) $H(\tilde{m}(z); y)$ is made by construction. They form a closed set under

$$\int_0^y dy \left( \frac{1}{y}, \frac{1}{1 - y}, \frac{1}{1 - y - z}, \frac{1}{y + z} \right) H(\tilde{b}(z); y)$$

2dHPL with up to three components in $\tilde{m}(z)$ can be expressed in terms of Nielsen’s polylogarithms. At four-component level:

one-dimensional integral representation

2dHPL are the basis functions for two-loop four-point functions with one off-shell leg

$\rightarrow$ computation of master integrals from differential equations reduces to algebraic determination of coefficients in the ansatz
Results

E. Remiddi, TG

All eight two-loop four-point master integrals corresponding to planar topologies have been computed from their differential equations

- Analytic expressions for all divergent parts
- One-dimensional integral representation for finite parts
- Confirm Smirnov’s recent result on two-loop double box with one off-shell leg

Work on non-planar master integrals is in progress
Results

For example:

\[
\begin{aligned}
&\frac{p_{123}}{p_1} = -\frac{1}{2}H(0, y)
\end{aligned}
\]

with:

\[
\begin{aligned}
f_{6.1.4}(y, z) &= 0 \\
f_{6.1.3}(y, z) &= -\frac{1}{2}H(0, y) \\
f_{6.1.2}(y, z) &= +H(0, 0; y) - H(1, 0; y) - \frac{\pi^2}{6} \\
f_{6.1.1}(y, z) &= +H(0; y)H(1, 0; z) + H(0; z)H(1 - z, 0; y) + H(0, 0; z)H(0; y) - 2H(0, 0; 0; y) \\
&- H(0, 1, 0; y) + H(0, 1, 0; z) + H(1, 0; z)H(1 - z; y) + 2H(1, 0, 0; y) - 2H(1, 1, 0; y) \\
&+ H(1, 1, 0; y) + H(1 - z, 1, 0; y) - 3\zeta_3 \\
&+ \frac{\pi^2}{6} [H(0; z) - 2H(1; y) + H(1; z) + H(1 - z; y)] \\
f_{6.1.0}(y, z) &= -3H(0; 0; 0, 0; z) - H(0; y)H(1, 1, 0; z) + 2H(0; z)H(1, 1 - z, 0; y) \\
&- 2H(0; z)H(1 - z, 0; y) - H(0; z)H(1 - z, 1 - z, 0; y) - 2H(0, 0; y)H(0, 0; z) \\
&- 2H(0, 0; y)H(1, 0; z) + 2H(0, 0; 0; z)H(1, 0; y) - 3H(0, 0; 0; z)H(1 - z, 0; y) \\
&- 3H(0, 0, 0; z)H(0; y) + 4H(0, 0, 0, 0; y) + 2H(0, 0, 1; 0; y) - 3H(0, 0, 1, 0; z) \\
&- H(0, 1, 0; y)H(0; y) + 2H(0, 0, 1; 0; z)H(1; y) - 3H(0, 1, 0; z)H(1 - z, 0; y) + 2H(0, 1, 1, 0; y) \\
&- 3H(0, 1, 0, 0; z) + H(0, 1, 1, 0; y) - 2H(0, 1 - z; y)H(1, 1; z) - 2H(0, 1 - z, 0; y)H(0; z) \\
&- 2H(0, 1 - z, 1, 0; y) + 2H(1, 0; y)H(0, 0; z) + 2H(1, 0; 0; z)H(1, 1 - z; y) \\
&- H(1, 1; z)H(1 - z, 0; y) - H(1, 0; z)H(1 - z, 1 - z; y) - 3H(1, 0, 0; z)H(1 - z; y) \\
&- 4H(1, 0, 0, 0; y) - 2H(1, 1, 0; y) - 3H(1, 1, 0; z) + 2H(1, 1, 0; z)H(1; y) \\
&+ 4H(1, 1, 0, 0; y) - 3H(1, 1, 0; 0; z) - 4H(1, 1, 1, 0; y) + 2H(1, 1 - z, 1, 0; y) \\
&- 2H(1 - z, 0, 1; 0; y) - 2H(1 - z, 1, 0; y) + 2H(1 - z, 1, 1; 0; y) - H(1 - z, 1 - z, 1, 0; y)
\end{aligned}
\]

\[
\begin{aligned}
f_{6.1.0}(y, z) &= \frac{\pi^4}{72} + \zeta_3 [-2H(0; y) - 3H(0; z) - 6H(1; y) - 3H(1; z) - 3H(1 - z; y)] \\
&+ \frac{\pi^2}{6} \left[ -H(0; y)H(0; z) - H(0; y)H(1; z) + 2H(0; z)H(1; y) - H(0; z)H(1 - z; y) \\
&- H(0, 0; z) + H(0, 1; y) - 2H(0, 1 - z; y) + 2H(1; y)H(1; z) - H(1, 0; z) - 4H(1, 1; y) \\
&+ 2H(1, 1 - z; y) - H(1 - z, 0; y) + 2H(1 - z, 1; y) - H(1 - z, 1 - z; y) \right]
\end{aligned}
\]
Summary and Outlook

Reduction of multi-leg integrals to a small set of master integrals using Integration-by-parts method and Lorentz-invariance

Calculation of master integrals using differential equations in external invariants

Solution of differential equations by construction of a linearly independent set of basis functions

Applications:

- Master integrals for massless two-loop four-point functions with one off-shell leg:
  
  $e^+e^- \rightarrow 3 \text{ Jets, } ep \rightarrow (2 + 1) \text{ Jets,}$
  
  $pp \rightarrow (W^\pm, Z^0, \gamma^*) + 1 \text{ Jet}$

  planar topologies: known
  non-planar topologies: work in progress