QCD Calculations by Numerical Integration

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Overview

• I propose a method for calculating infrared safe observables at next-to-leading order in field theories.
• The application so far is to QCD jet cross sections *etc.* in e^+e^- annihilation.
• See Phys. Rev. D 62, 014009 (2000)
and Phys. Rev. Lett. 81, 2638 (1998).
• A program, *beowulf*, with documentation is available at http://zebu.uoregon.edu/~soper/beowulf/

• It should be possible to extend the method to
$e + p \to e + jets + X$.
$\bar{p} + p \to jets + X$.

• We do not need analytical calculations of loop diagrams.
• Thus the method is very flexible.

• For example, adding masses, $SU(2) \times U(1)$ interactions, SUSY interactions, ...should be straightforward.

• The main idea is simple.
• Anyone can do it.

e^+ e^- annihilation event shape observables

• Existing program works for infrared-safe, three-jet-like observables in electron-positron annihilation at order α_s^2.

• Examples: thrust distribution, three jet cross section.

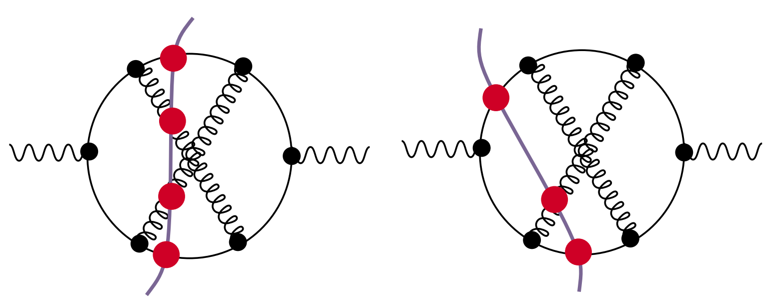
• Ingredients: Feynman diagrams, measurement functions $\mathcal{S}_3(\vec{k}_1, \vec{k}_2, \vec{k}_3)$ and $\mathcal{S}_4(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4)$.

$$\sigma = \frac{1}{3!} \int d\vec{k}_1 d\vec{k}_2 d\vec{k}_3 \; \frac{d\sigma_3}{d\vec{k}_1 d\vec{k}_2 d\vec{k}_3} \; \mathcal{S}_3(\vec{k}_1, \vec{k}_2, \vec{k}_3)$$
$$+ \frac{1}{4!} \int d\vec{k}_1 d\vec{k}_2 d\vec{k}_3 d\vec{k}_4 \; \frac{d\sigma_4}{d\vec{k}_1 d\vec{k}_2 d\vec{k}_3 d\vec{k}_4} \; \mathcal{S}_4(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4).$$

The functions $\mathcal{S}$ are infrared safe:

$$\mathcal{S}_4(\vec{k}_1, \vec{k}_2, \lambda\vec{k}_3, (1-\lambda)\vec{k}_3) = \mathcal{S}_3(\vec{k}_1, \vec{k}_2, \vec{k}_3).$$

• Think of $\mathcal{I}$ as computed from cut Feynman diagrams.



• The red dots indicate the measurement functions $\mathcal{S}$.

Does it work?

Comparison of results for moments of the thrust distribution,

$$\mathcal{I}_n = \frac{1}{\sigma_0\,(\alpha_s/\pi)^2}\int_0^1 dt\;(1-t)^n\,\frac{d\sigma^{[2]}}{dt},$$

where $d\sigma^{[2]}/dt$ is the $(\alpha_s/\pi)^2$ contribution to $d\sigma/dt$. The "numerical" results are from the program *beowulf*. The first error is statistical, the second systematic. The "numerical/analytical" results are from the program of Kunszt and Nason and are given with their reported statistical errors.

n	numerical	numerical/analytical
1.5	$4.127 \pm 0.008 \pm 0.025$	4.132 ± 0.003
2.0	$1.565 \pm 0.002 \pm 0.007$	1.565 ± 0.001
2.5	$(6.439 \pm 0.010 \pm 0.022) \times 10^{-1}$	$(6.440 \pm 0.003\ ) \times 10^{-1}$
3.0	$(2.822 \pm 0.005 \pm 0.009) \times 10^{-1}$	$(2.822 \pm 0.001\ ) \times 10^{-1}$
3.5	$(1.296 \pm 0.002 \pm 0.004) \times 10^{-1}$	$(1.296 \pm 0.0005) \times 10^{-1}$
4.0	$(6.159 \pm 0.011 \pm 0.016) \times 10^{-2}$	$(6.161 \pm 0.002\ ) \times 10^{-2}$
4.5	$(3.009 \pm 0.006 \pm 0.007) \times 10^{-2}$	$(3.010 \pm 0.0006) \times 10^{-2}$
5.0	$(1.501 \pm 0.003 \pm 0.003) \times 10^{-2}$	$(1.502 \pm 0.0002) \times 10^{-2}$

Comparison of results for moments of the y_{cut} distribution. Let $f_3(y_{\rm cut})$ be the fraction of events that have 3 jets and let $g_3(y_{\rm cut})$ be the derivative

$$g_3(y_{\rm cut}) = -\frac{f_3(y_{\rm cut})}{dy_{\rm cut}}.$$

Then define

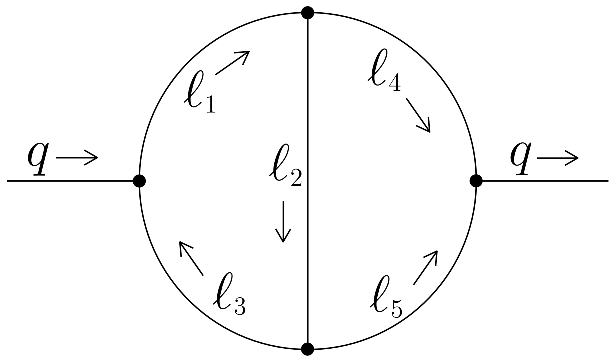
$$\mathcal{I}_n = \frac{1}{\sigma_0\,(\alpha_s/\pi)^2}\int_0^1 d\,y_{\rm cut}\ (y_{\rm cut})^n g_3^{[2]}.$$

The "numerical" results are from the program *beowulf.* The first error is statistical, the second systematic. The "numerical/analytical" results are from the program of Kunszt and Nason and are given with their reported statistical errors.

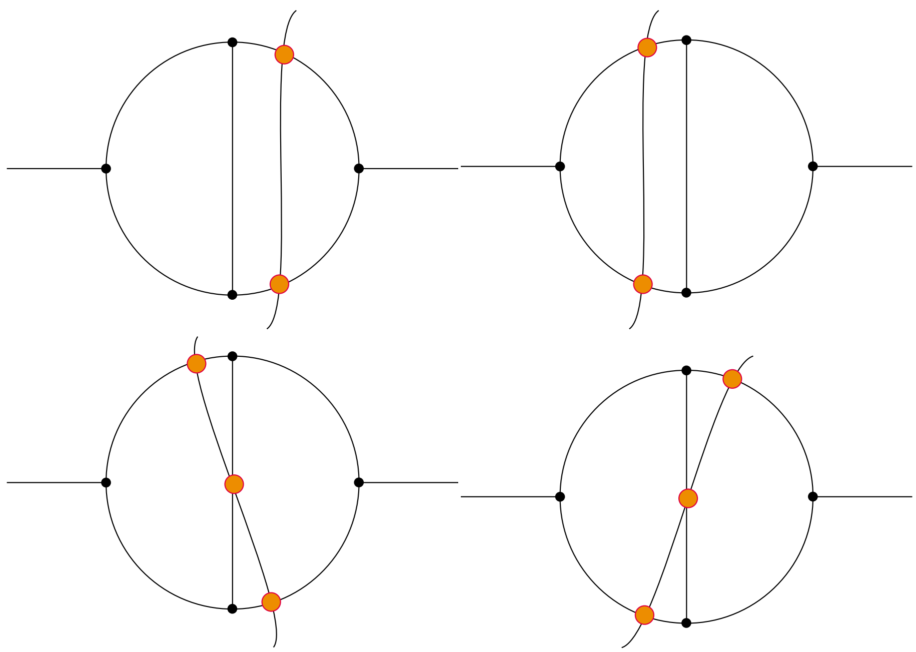
n	numerical	numerical/analytical
1.5	$(8.442 \pm 0.034 \pm 0.059) \times 10^{-1}$	$(8.397 \pm 0.002\ ) \times 10^{-1}$
2.0	$(3.106 \pm 0.012 \pm 0.015) \times 10^{-1}$	$(3.090 \pm 0.0004) \times 10^{-1}$
2.5	$(1.205 \pm 0.005 \pm 0.005) \times 10^{-1}$	$(1.200 \pm 0.0002) \times 10^{-1}$
3.0	$(4.945 \pm 0.025 \pm 0.019) \times 10^{-2}$	$(4.927 \pm 0.001\ ) \times 10^{-2}$
3.5	$(2.122 \pm 0.012 \pm 0.008) \times 10^{-2}$	$(2.116 \pm 0.0007) \times 10^{-2}$
4.0	$(9.430 \pm 0.064 \pm 0.032) \times 10^{-3}$	$(9.412 \pm 0.004\ ) \times 10^{-3}$
4.5	$(4.304 \pm 0.034 \pm 0.014) \times 10^{-3}$	$(4.301 \pm 0.002\ ) \times 10^{-3}$
5.0	$(2.008 \pm 0.018 \pm 0.006) \times 10^{-3}$	$(2.008 \pm 0.001\ ) \times 10^{-3}$

Example for this talk

Consider a two loop diagram in ϕ^3 theory.



There are four cuts.

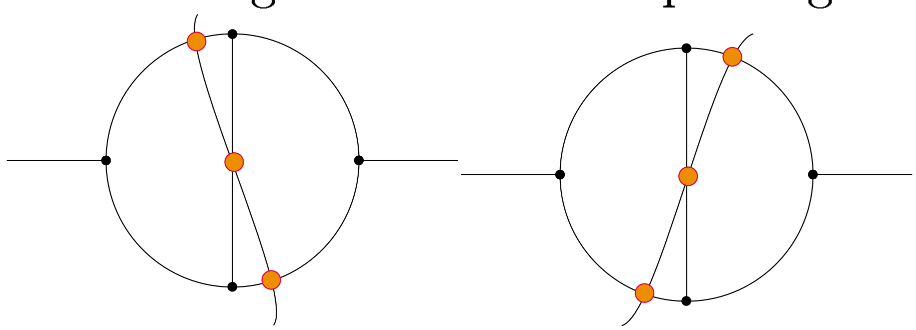


- Fix $\vec{q}$ and integrate over q^0.
- $\mathcal{S}(\ell) = \sum |\vec{k}_{T,i}|$, where $\vec{k}_T \cdot \vec{q} = 0$.

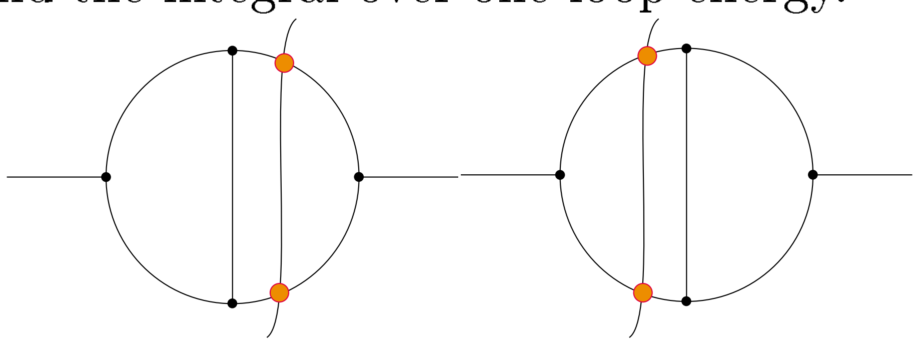
Energy integrals for final state partons

- For each final state parton, we have $\delta(E^2 - \vec{k}^2)\,\Theta(E > 0)$.
- Thus, $E = |\vec{k}|$.

- With three final state particles, we eliminate the integral over q^0 and the integrals over two loop energies.



- With two final state particles, we eliminate the integral over q^0 and the integral over one loop energy.



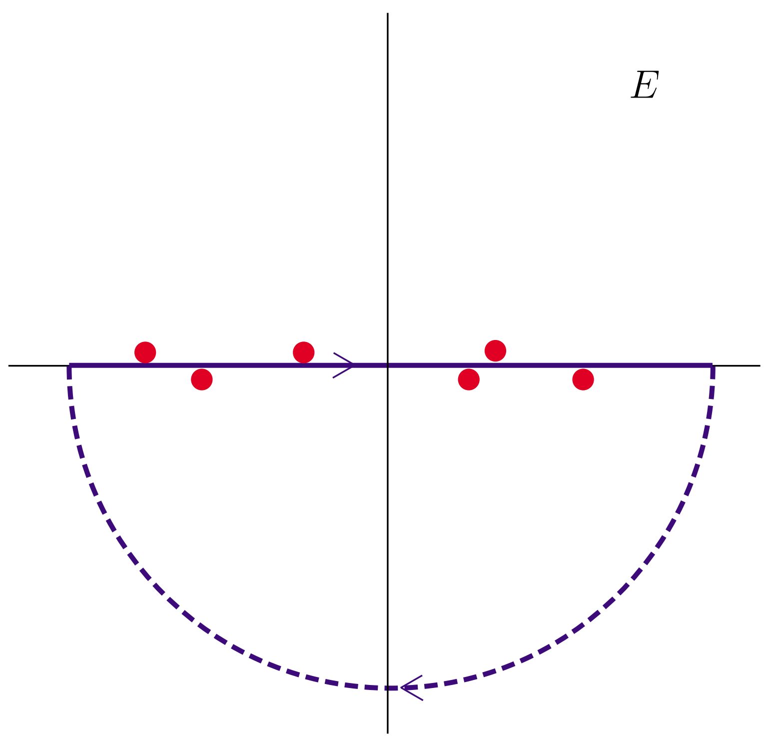
- One integral over the energy in a virtual loop remains.

Energy integrals for a virtual loop

For virtual loop, we have a structure

$$\int \frac{dE}{2\pi}\,\mathcal{N}(E)\;\frac{i}{(E-|\vec{k}|+i\epsilon)(E+|\vec{k}|-i\epsilon)}$$
$$\times\,\frac{i}{(E-Q_1^0-|\vec{k}-\vec{Q}_1|+i\epsilon)(E-Q_1^0+|\vec{k}-\vec{Q}_1|-i\epsilon)}$$
$$\times\,\frac{i}{(E-Q_2^0-|\vec{k}-\vec{Q}_2|+i\epsilon)(E-Q_2^0+|\vec{k}-\vec{Q}_2|-i\epsilon)}\cdots$$

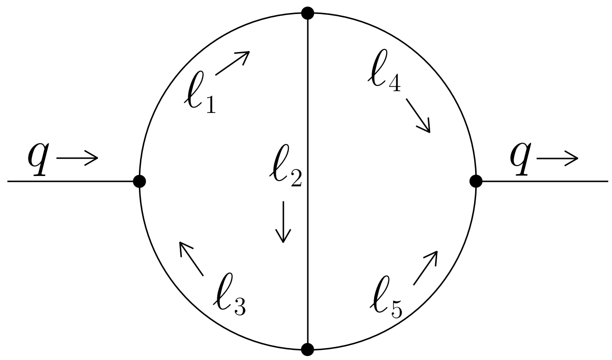
- For a virtual loop, do this by "closing the contour."



- This gives successive $E-Q_i^0=|\vec{k}-\vec{Q}_i|$ substitutions.

- The operation is algebraic.

The integration



• Group together the contributions from all of the cuts C of the Feynman graph. This gives

$$\mathcal{I} = \int d^3\vec{\ell}_4 \, d^3\vec{\ell}_2 \sum_C g(C;\ell)$$

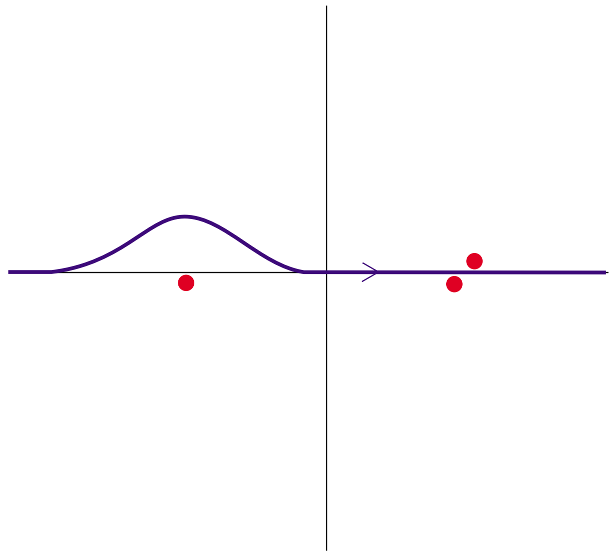
This talk:

• Fix $\vec{\ell}_4$.

• Examine (real part of) integrand versus $\vec{\ell} \equiv \vec{\ell}_2$ in plane of $\vec{q}$ and $\vec{\ell}_4$.

Contour deformation

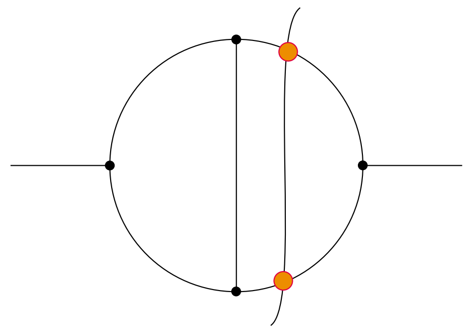
Beware! The integrands contain singularities.



We must distinguish between *pinch singular points* and singularities that are not pinched. (Sterman, 1978).

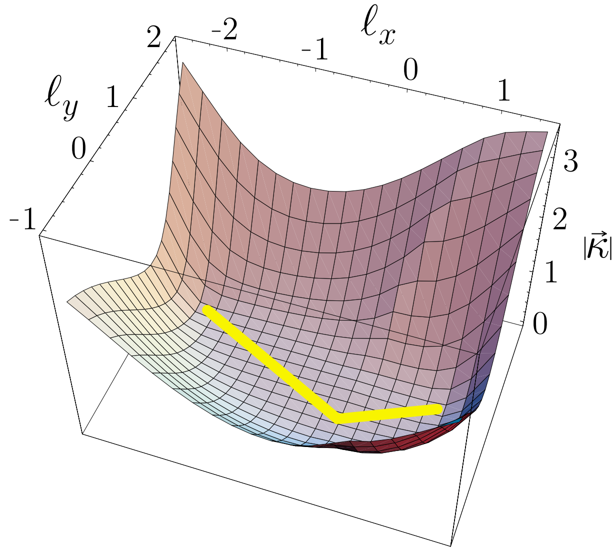
• We will deform the integration contour away from singularities that do not pinch it.

Contour deformation



Call the loop momentum on the deformed contour $\vec{\ell} + i\vec{\kappa}$.

Here is a choice for $|\vec{\kappa}|$ for the cut shown above.



$|\vec{\kappa}|$ vs. ℓ_x, ℓ_y

This gives

$$\mathcal{I} = \int d^3\vec{\ell}_4 \, d^3\vec{\ell}_2 \sum_C \mathcal{J}(C;\ell) \; g(C;\ell + i\kappa(C;\ell))$$

- The integral is convergent.
- Calculate it by Monte Carlo integration.

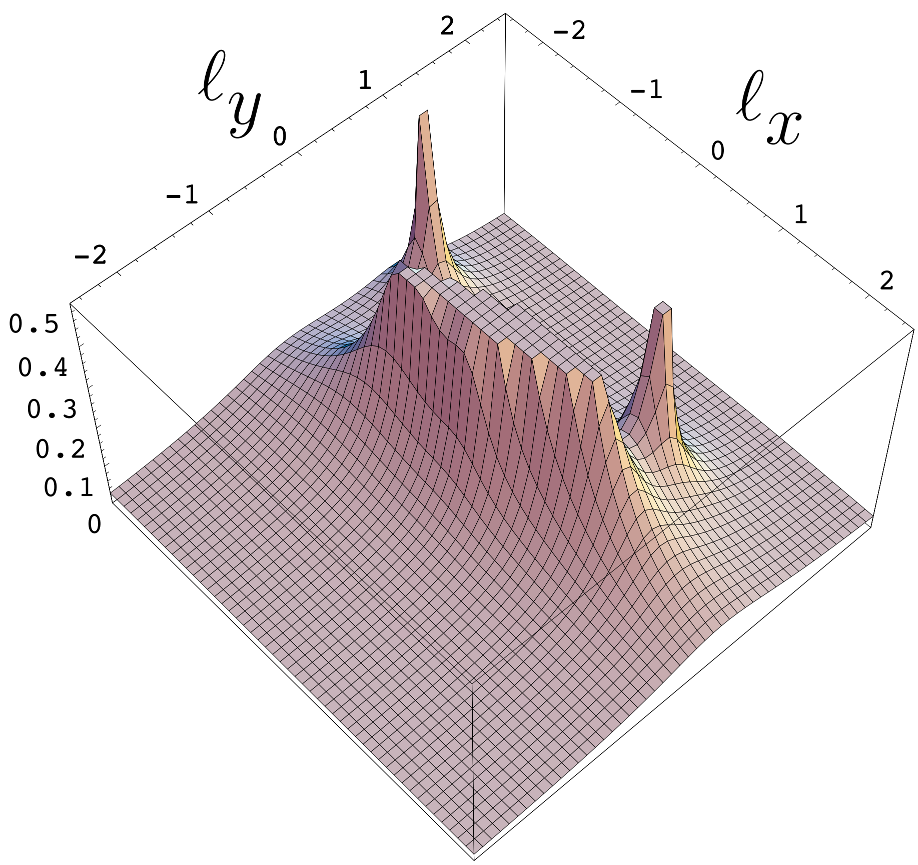
Monte Carlo integration

$$\mathcal{I} = \int d\ell \; f(\ell) \approx \frac{1}{N} \sum_i \frac{f(\ell_i)}{\rho(\ell_i)}$$

- To minimize the errors, make sure density

$$\frac{|f(\ell)|}{\rho(\ell)}$$

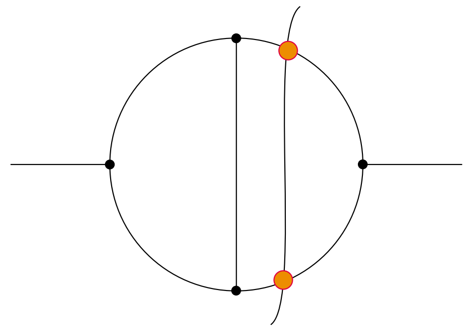
is never too large. Here is a choice for ρ.



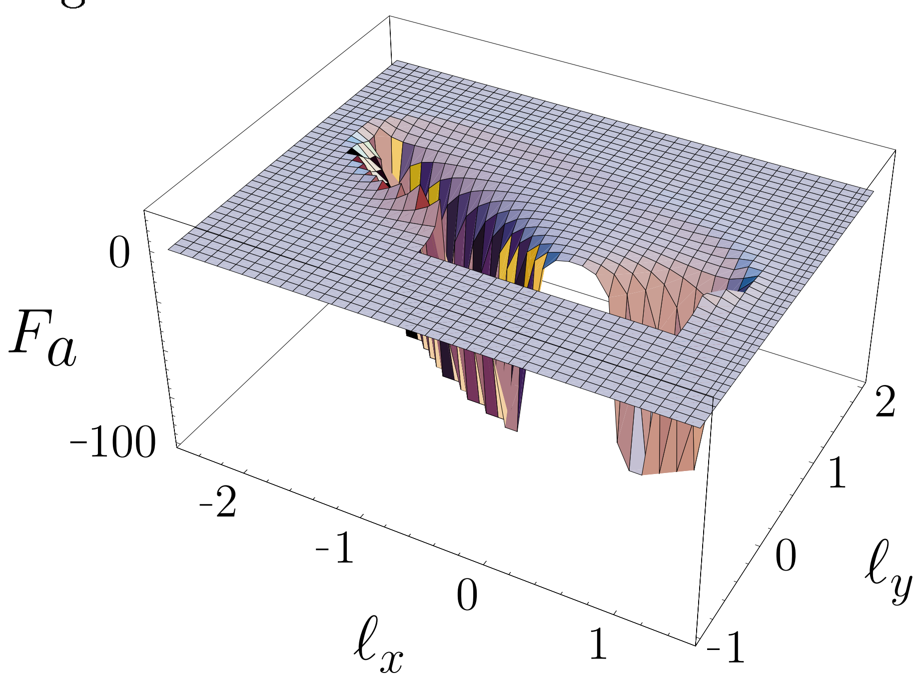
ρ vs. ℓ_x, ℓ_y

Following: plots of $F \equiv f/\rho$.

Right cut

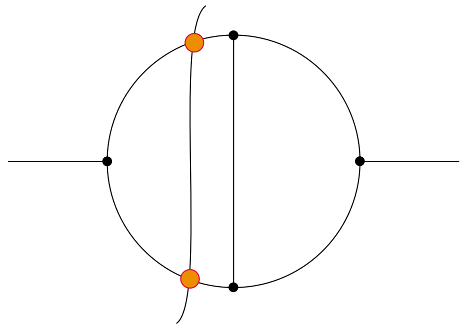


• The contribution from the virtual graph has collinear and soft singularities.

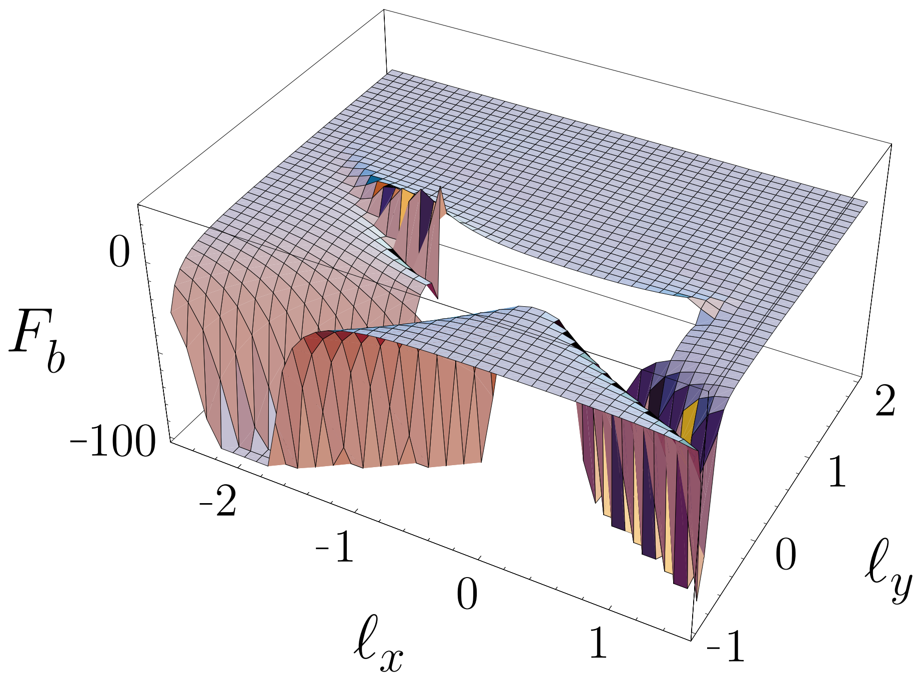


• In the Ellis-Ross-Terrano method (1981) the virtual subdiagram is calculated analytically and these singularities require careful attention.

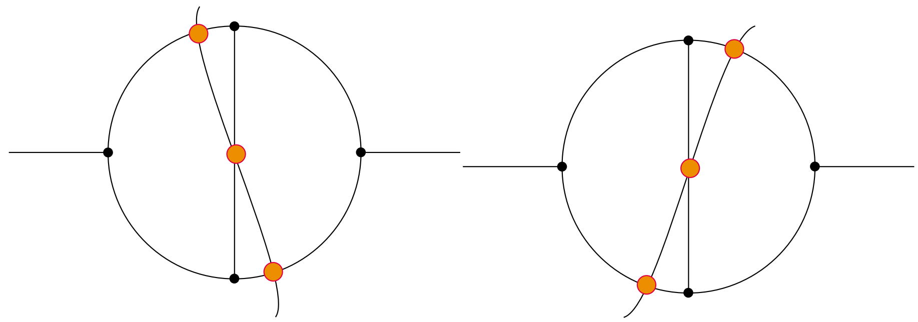
Left cut



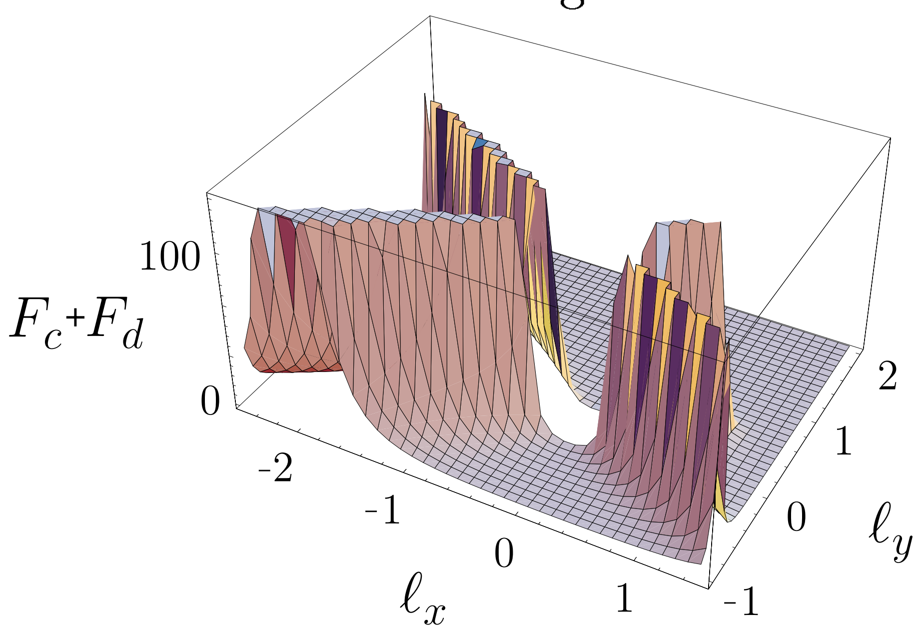
• The other cut graph with a virtual subgraph is also singular.



Cuts with no virtual loop

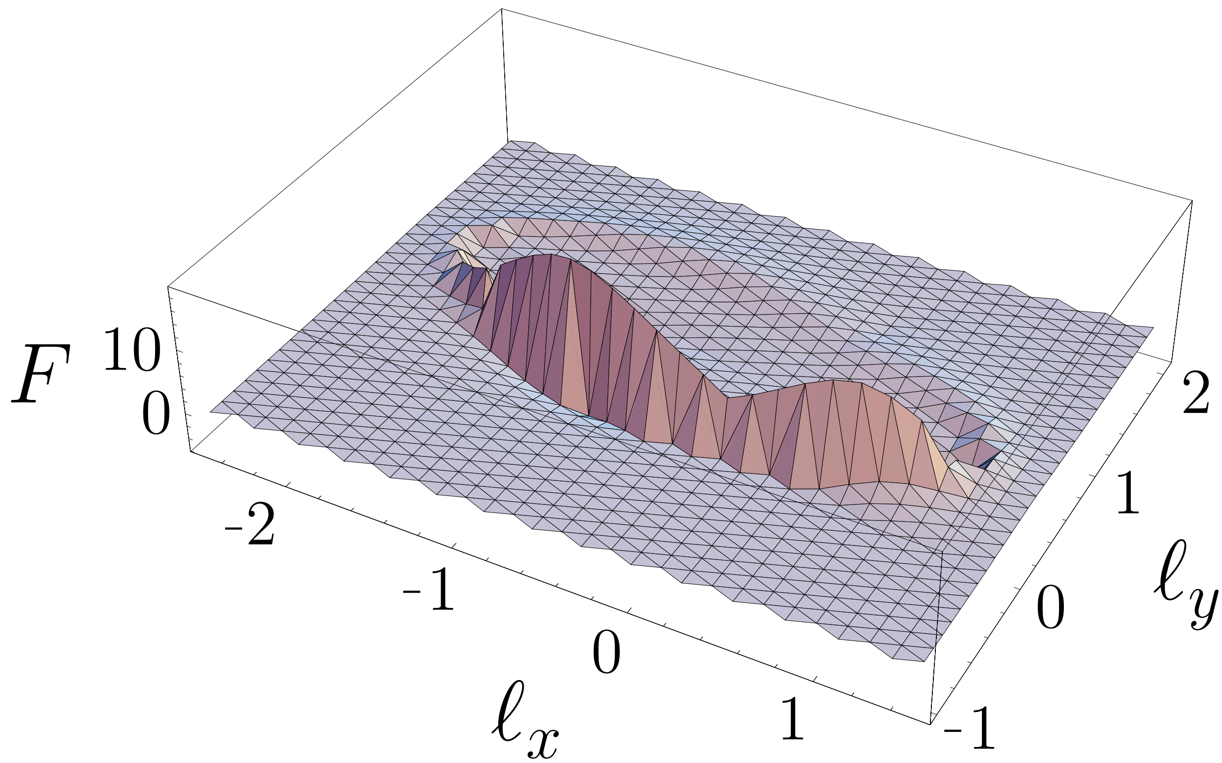


• There are collinear and soft singularities.



• In the Ellis-Ross-Terrano method, these graphs would be calculated using a numerical integration. But first a cutoff or other method for eliminating the singularities is needed.

In the sum, the singularities cancel



• In the completely numerical method, we add the contributions first, then integrate.

• The integrand f has integrable singularities at points where one parton momentum vanishes, but we have chosen the density of points ρ so that f/ρ is everywhere finite.

Wish list

$e^+ + e^- \to 3$ jets **at NLO**

- Add masses.
- Add interactions: $SU(2) \times U(1)$, SUSY.

Add initial state hadrons

- $e + p \to e + 2$ jets $+ X$ at NLO.
- $p + p \to 2$ jets $+ X$ at NLO.
- Add masses.
- Add interactions: $SU(2) \times U(1)$, SUSY.

Add more partons

- $e^+ + e^- \to 4$ jets at NLO.
- $p + p \to 3$ jets $+ X$ at NLO.
- Add masses.
- Add interactions: $SU(2) \times U(1)$, SUSY.

$e^+ + e^- \to 3$ jets **at NNLO**

Go beyond fixed order perturbation theory

- Bubble sums /running coupling for power corrections.
- Add leading log parton showers to NLO calculation.

- Preliminary step:
  $e^+ + e^- \to 3$ jets at NLO in Coulomb gauge.
  Work underway by DES and M. Krämer (Edinburgh).